

BRST QUANTIZATION OF QUASI-SYMPLECTIC MANIFOLDS AND BEYOND

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ABSTRACT. We consider a class of *factorizable* Poisson brackets which includes almost all reasonable Poisson structures. A particular case of the factorizable brackets are those associated with symplectic Lie algebroids. The BRST theory is applied to describe the geometry underlying these brackets as well as to develop a deformation quantization procedure in this particular case. This can be viewed as an extension of the Fedosov deformation quantization to a wide class of *irregular* Poisson structures. In a more general case, the factorizable Poisson brackets are shown to be closely connected with the notion of n -algebroid. A simple description is suggested for the geometry underlying the factorizable Poisson brackets basing on construction of an odd Poisson algebra bundle equipped with an abelian connection. It is shown that the zero-curvature condition for this connection generates all the structure relations for the n -algebroid as well as a generalization of the Yang-Baxter equation for the symplectic structure.

1. INTRODUCTION

The deformation quantization of a Poisson manifold $(M, \{\cdot, \cdot\})$ is the construction of a local one-parameter deformation of the commutative algebra of functions $C^\infty(M)$ respecting associativity [1], [2]. The deformed product is usually denoted by $*$, and the deformation parameter is the Plank constant \hbar . In each order in \hbar the $*$ -product is given by a bi-differential operator (locality) and the skew-symmetric part of the first \hbar -order coincides with the Poisson bracket of functions (correspondence principle).

Very early it appeared that the complexity of the deformation quantization program essentially depends on whether a given Poisson manifold is regular or not. In the regular case,

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i.e., where the rank of the Poisson tensor is constant, one can introduce an affine symmetric connection respecting the Poisson structure (a Poisson connection). Clearly, in the irregular case such a connection cannot exist. The relevance of the Poisson connection for constructing $*$ -products had been already discussed in [2], but in its full strength, the connection was first exploited by Fedosov in his seminal paper [3] on the deformation quantization of symplectic and regular Poisson manifolds (see also [4]).

The existence of deformation quantization for general Poisson manifolds, not necessarily regular, was proved by Kontsevich [5] as a consequence of his Formality Theorem. An explicitly covariant version of the Kontsevich quantization has been given in [6] (see also [7], where both covariant and equivariant versions of the formality theorem have been presented). It should be noted that the Kontsevich quantization is based on completely different ideas and involves more complicated algebraic technique as compared to the Fedosov quantization. A nice “physical explanation” of the Kontsevich quantization formula was given in [8] by applying the BV quantization method [9] to the Poisson sigma-model.

Recently, it was recognized that the method of Fedosov’s quantization can further be extended to include a certain class of irregular Poisson manifolds even though no Poisson connection can exist in this case. To give an idea about the manifolds in question let us write the following expression describing the general structure of the corresponding Poisson brackets:

$$(1.1) \quad \{f, g\} = \omega^{ab}(X_a^\mu \partial_\mu f)(X_b^\nu \partial_\nu g), \quad \det(\omega^{ab}) \neq 0.$$

The matrices X and ω are subject to certain conditions ensuring the Jacobi identity. The geometric meaning of these conditions as well as the precise mathematical status of X and ω will be explained in the next section. Here we would like to mention that, no a priori assumption is made about the rank of the matrix X , so the Poisson brackets (1.1) may well be irregular.

In the case where the matrix X is the anchor of a Lie algebroid the manifolds under consideration are something intermediate between symplectic and general Poisson manifolds. For this reason, we refer to them as *quasi-symplectic* Poisson manifolds (not to be confused with the quasi-Poisson manifolds introduced in Ref. [10]). Being closely related with the notion of a

dynamical r -matrix, these manifolds may be of immediate interest in the theory of integrable systems.

The generalization of the Fedosov deformation quantization to the case of symplectic Lie algebroids was first given by Nest and Tsygan [11]. They also proved corresponding classification theorems. Fedosov's quantization method was also described in the work [12] for the same class of manifolds in the language of symplectic ringed spaces. Particular classes of quasi-symplectic manifolds have been quantized in [13], [14], [15] making use of various ideas, including BRST theory.

The aim of this work is twofold. In the first part of the paper we put the deformation quantization of quasi-symplectic manifolds in the framework of BFV-BRST theory [16], [17], [18]. For the (constrained) Hamiltonian systems on symplectic manifolds, the relationship has been already established between the BFV-BRST and the Fedosov quantizations [19],[20]. Here we re-shape this technology to make it working in a more general case of quasi-symplectic manifolds. The second part of the paper is devoted to a possible generalization of the notion of a quasi-symplectic manifold to the case of n -algebroids or, in other terminology, NQ-manifolds [21], [22], [23], [24]. This generalization essentially relaxes the restrictions on the structure functions X and ω , entering factorization (1.1), and covers almost all reasonable Poisson structures.

The paper is organized as follows. In Sect.2 we give the definition of a quasi-symplectic Poisson manifold and discuss some examples. Here we also construct a simple counter-example to existence of a quasi-symplectic representation for any Poisson bracket. Sect.3 deals with realization of quasi-symplectic manifolds as coisotropic surfaces in the total space of vector bundles associated with symplectic Lie algebroids. In Sect.4 this realization is exploited to perform the BRST quantization of the resulting gauge system. We prove that the quantum multiplication in the algebra of physical observables induces an associative $*$ -product on the initial quasi-symplectic manifold. In Sect.5 we generalize the notion of a quasi-symplectic manifold to a wider class of factorizable Poisson brackets. Under reasonable restrictions this class of Poisson structures is proved to be closely connected with n -algebroids. Using the 2-algebroid as example, we show how the geometry underlying factorizable Poisson brackets can

be described in terms of a super-vector bundle equipped with a fiber-wise odd Poisson structure and a compatible abelian connection.

2. QUASI-SYMPLECTIC MANIFOLDS: DEFINITION AND EXAMPLES

The most concise and geometrically transparent way to define the quasi-symplectic manifolds is to use the notion of a *Lie algebroid* [25].

Definition. A Lie algebroid over a manifold M is a (real) vector bundle $\mathcal{E} \rightarrow M$ equipped with the following additional structures.

- (1) There is a (real) Lie algebra structure on the linear space of sections $\Gamma(\mathcal{E})$.
- (2) There is a bundle map $\rho : \mathcal{E} \rightarrow TM$ such that the Lie algebra and $C^\infty(M)$ -module structures on $\Gamma(\mathcal{E})$ are compatible in the following sense:

$$(2.1) \quad [s_1, f s_2] = f[s_1, s_2] + (\rho_*(s_1)f)s_2, \quad \forall f \in C^\infty(M), \quad \forall s_1, s_2 \in \Gamma(\mathcal{E}).$$

The map ρ is called the *anchor* of the Lie algebroid $\mathcal{E} \rightarrow M$.

The last relation can be viewed as the Leibniz rule for the Lie algebroid bracket. Using this relation and the Jacobi identity for the bracket it is not hard to see that the anchor map $\rho : \mathcal{E} \rightarrow TM$ defines a Lie algebra homomorphism on sections, i.e.,

$$(2.2) \quad \rho_*([s_1, s_2]) = [\rho_*(s_1), \rho_*(s_2)], \quad \forall s_1, s_2 \in \Gamma(\mathcal{E}),$$

where the brackets in the r.h.s. stand for the commutator of vector fields.

It is instructive to look at the local coordinate expression of the above relations. Let x^μ be a coordinate system on a trivializing chart $\mathcal{U} \subset M$ and let s_a be a frame of $\mathcal{E}|_{\mathcal{U}}$. By definition, we have

$$(2.3) \quad [s_a, s_b] = f_{ab}^c(x)s_c, \quad \rho_*(s_a) = X_a^\mu(x)\partial_\mu,$$

$$\mu = 1, \dots, \dim M, \quad a = 1, \dots, \text{rank } \mathcal{E}.$$

In view of Rels. (2.2) and (2.1) the structure functions $f_{ab}^c, X_a^\mu \in C^\infty(U)$ meet the following conditions:

$$(2.4) \quad [X_a, X_b]^\nu := X_a^\mu \partial_\mu X_b^\nu - X_b^\mu \partial_\mu X_a^\nu = f_{ab}^c X_c^\nu,$$

$$(2.5) \quad f_{ab}^d f_{dc}^e - X_c^\mu \partial_\mu f_{ab}^e + \text{cycle}(a, b, c) = 0.$$

Notice that the second relation is automatically satisfied for any vector bundle \mathcal{E} of rank 1 or 2, whereas in the case of $\text{rank } \mathcal{E} > 2$ it becomes an actual restriction on the structure functions f_{ab}^c .

In general, $\rho(\mathcal{E})$ is not a smooth subbundle of TM as the rank of the distribution $\rho(\mathcal{E})$ may vary from point to point. Nonetheless, in view of (2.2), $\rho(\mathcal{E})$ generates a (singular) integrable distribution in the sense of Sussman [26]: for each $p \in M$ there is a smooth submanifold $\Sigma_p \subset M$ such that $p \in \Sigma_p$ and $T_q \Sigma_p = \rho(\mathcal{E}_q)$ for any $q \in \Sigma_p$. The corresponding foliation will be denoted by $F(M)$.

Example. Any tangent bundle TM may be viewed as a Lie algebroid with the Lie bracket given by the commutator of vector fields and the anchor $\rho = \text{id} : TM \rightarrow TM$.

2.1. Differential geometry of Lie algebroids. One can regards the concept of a Lie algebroid as a tool for transferring all the usual differential-geometric constructions from a tangent bundle to an abstract vector bundle. In particular, it is possible to define the Lie algebroid counterpart of the exterior calculus. Denote by $\Lambda(\mathcal{E}) = \bigoplus \Lambda^p(\mathcal{E})$ the exterior algebra of sections $\Gamma(\wedge^\bullet \mathcal{E}^*)$, \mathcal{E}^* being the bundle dual to \mathcal{E} . Consider the following nilpotent operator $d : \Lambda^p(\mathcal{E}) \rightarrow \Lambda^{p+1}(\mathcal{E})$:

$$(2.6) \quad \begin{aligned} d\alpha(s_0, \dots, s_p) &= \sum_{k=0}^p (-1)^k \rho_*(s_k)(\alpha(s_0, \dots, \hat{s}_k, \dots, s_p)) \\ &+ \sum_{k < n=1}^p (-1)^{k+n} \alpha([s_k, s_n], s_0, \dots, \hat{s}_k, \dots, \hat{s}_n, \dots, s_p), \end{aligned}$$

for all $s_0, s_1, \dots, s_p \in \Gamma(\mathcal{E})$. Since $d^2 = 0$, we have a generalization of the De Rham complex. We will refer to elements of $\Lambda^p(\mathcal{E})$ as \mathcal{E} - p -forms, or just p -forms when it cannot lead to confusion. Note that $\Lambda^0(\mathcal{E})$ is naturally identified with $C^\infty(M)$.

More generally, one may consider the tensor product $\mathcal{E} \otimes V$, where $V \rightarrow M$ is a vector bundle with connection ∇ . Then, ∇ induces the covariant derivative $\nabla_\rho : \Lambda^p(\mathcal{E}, V) \rightarrow \Lambda^{p+1}(\mathcal{E}, V)$ on the space $\Lambda(M, \mathcal{E}) = \Lambda(\mathcal{E}) \otimes \Gamma(V)$ of $\Gamma(V)$ -valued \mathcal{E} -forms:

$$(2.7) \quad \begin{aligned} \nabla_\rho \omega(s_0, \dots, s_p) &= \sum_{k=0}^p (-1)^k \nabla_{\rho_*(s_k)} (\omega(s_0, \dots, \hat{s}_k, \dots, s_p)) \\ &+ \sum_{k < n=1}^p (-1)^{k+n} \alpha([s_k, s_n], s_0, \dots, \hat{s}_k, \dots, \hat{s}_n, \dots, s_p). \end{aligned}$$

The curvature of ∇_ρ is defined in the usual way:

$$(2.8) \quad R = \nabla_\rho^2 : \Gamma(V) \rightarrow \Lambda^2(\mathcal{E}, V).$$

One may verify that

$$(2.9) \quad R(fu) = fRu, \quad \forall f \in C^\infty(M), \quad \forall u \in \mathcal{E},$$

so that in each coordinate chart the curvature R is given by a matrix valued 2-form determining a $C^\infty(M)$ -linear automorphism of $\Gamma(V)$. Like the curvature of the bundle connection ∇ , R satisfies the *Bianchi identity*

$$(2.10) \quad [\nabla_\rho, \nabla_\rho^2] = 0 \Leftrightarrow \nabla_\rho R = 0.$$

(To write the last formula we extend the action of ∇ from V to the tensor product $V \otimes V^*$ by the usual formulas of differential geometry.)

In what follows we will mostly deal with the case $V = \mathcal{E}$. Then, in addition to the curvature, one more covariant of the connection can be introduced. The *torsion* T of a Lie algebroid connection ∇_ρ is an element of $\Lambda^2(\mathcal{E}, \mathcal{E})$ defined by the rule

$$(2.11) \quad \Gamma(\mathcal{E}) \ni T(u, v) = \nabla_{\rho_*(u)} v - \nabla_{\rho_*(v)} u - [u, v], \quad \forall u, v \in \Gamma(\mathcal{E}).$$

If $\Gamma_{\mu b}^a$ are coefficients of the connection ∇ with respect to local coordinates x^μ and a frame s_a , then the components of the torsion tensor read

$$(2.12) \quad T_{ab}^c = X_a^\mu \Gamma_{\mu b}^c - X_b^\mu \Gamma_{\mu a}^c - f_{ab}^c.$$

The components of the curvature tensor R are

$$(2.13) \quad R_{abc}^d = X_a^\mu X_b^\nu R_{\mu\nu c}^d,$$

where

$$(2.14) \quad R_{\mu\nu c}^d = \partial_\mu \Gamma_{\nu c}^d - \partial_\nu \Gamma_{\mu c}^d + \Gamma_{\mu a}^d \Gamma_{\nu c}^a - \Gamma_{\nu a}^d \Gamma_{\mu c}^a$$

is the curvature of ∇ . There is a simple formula relating the exterior and covariant derivatives:

$$(2.15) \quad \begin{aligned} d\alpha(s_0, \dots, s_p) &= \sum_{k=0}^p (-1)^k (\nabla_{\rho_*(s_k)} \alpha)(s_0, \dots, \hat{s}_k, \dots, s_p) \\ &+ \sum_{k < n=1}^p (-1)^{k+n} \alpha(T(s_k, s_n), s_0, \dots, \hat{s}_k, \dots, \hat{s}_n, \dots, s_p); \end{aligned}$$

here we use the isomorphism $\Lambda^p(\mathcal{E}) \simeq \Lambda^0(\mathcal{E}, \wedge^p \mathcal{E})$. A straightforward computation yields the *torsion Bianchi identity*

$$(2.16) \quad \nabla_c T_{ab}^d + R_{abc}^d + \text{cycle}(a, b, c) = 0,$$

where $\nabla_a := \nabla_{\rho_*(s_a)}$.

In this paper we are interested in the Lie algebroids endowed with a closed and non-degenerate 2-form $\omega \in \Lambda^2(\mathcal{E})$. A 2-form ω is called non-degenerate if the equality

$$(2.17) \quad \omega(u, v) = 0, \quad \forall u \in \Gamma(\mathcal{E}),$$

implies $v = 0$. In terms of local coordinates the closedness condition $d\omega = 0$ reads

$$(2.18) \quad X_c^\mu \partial_\mu \omega_{ab} + \omega_{cd} f_{ab}^d + \text{cycle}(a, b, c) = 0,$$

where $\omega_{ab} := \omega(s_a, s_b)$. Extending the analogy with classical differential geometry, we refer to ω as the *symplectic form* and call the triple $(\mathcal{E}, \rho, \omega)$ the *symplectic Lie algebroid*¹.

¹This is a particular example of *triangular Lie bialgebroids* studied in Ref. [27].

2.2. Quasi-symplectic manifolds. It is well known that any symplectic structure on a Lie algebroid $\mathcal{E} \rightarrow M$ gives rise to a Poisson structure on the base manifold M . It is this Poisson structure we are going to quantize by the BFV-BRST method.

Proposition 2.1. *Let $(\mathcal{E}, \rho, \omega)$ be a symplectic Lie algebroid, then M is a Poisson manifold w.r.t. to the following Poisson bracket:*

$$(2.19) \quad \{f, g\} = \omega^{-1}(df, dg), \quad \forall f, g \in C^\infty(M) \simeq \Lambda^0(\mathcal{E});$$

here ω^{-1} is the bi-section inverse to the symplectic form $\omega \in \Lambda^2(\mathcal{E})$ and $df, dg \in \Lambda^1(\mathcal{E})$ are the differentials defined by (2.6).

Proof. In terms of local coordinates the Poisson bi-vector determining the bracket (2.19) has the form

$$(2.20) \quad \alpha = \omega^{ab} X_a \wedge X_b \in \wedge^2 TM, \quad \{f, g\} = \alpha^{\mu\nu} \partial_\mu f \partial_\nu g,$$

where $X_a := X_a^\mu \partial_\mu$, $\omega^{ac} \omega_{cb} = \delta_b^a$, and $\alpha^{\mu\nu} = \omega^{ab} X_a^\mu X_b^\nu$. The Jacobi identity for α follows immediately from the Lie algebroid relations (2.4) and the closedness condition (2.18). Indeed, using the Leibniz rule for the Schouten bracket of α with itself, we get

$$(2.21) \quad \begin{aligned} \frac{1}{4}[\alpha, \alpha] &= \omega^{ab} \omega^{cd} X_a \wedge [X_b, X_c] \wedge X_d + \omega^{ab} [X_b, \omega^{cd}] X_a \wedge X_c \wedge X_d \\ &= (\omega^{am} f_{mn}^b \omega^{nc} + \omega^{am} X_m^\mu \partial_\mu \omega^{bc}) X_a \wedge X_b \wedge X_c = (d\omega)_{abc} X^a \wedge X^b \wedge X^c = 0. \end{aligned}$$

The indices are lowered and raised with the help of the symplectic form ω and its inverse.

Since the rank of the anchor distribution may vary through M , the induced Poisson structure (2.20) is irregular in general (though it involves a non-degenerate bi-vector $\omega^{-1} \in \wedge^2 \mathcal{E}$). For this reason and following the terminology of the work [12] we refer to (M, α) as the *quasi-symplectic manifold*. Accordingly, Rel.(2.19) is said to define a *quasi-symplectic representation* for the Poisson bi-vector α .

Given a symplectic Lie algebroid, we have two (singular) foliations on M : the anchor foliation $F(M)$ and the symplectic foliation $S(M)$ associated with the induced Poisson structure (2.20).

Clearly, the latter foliation is subordinated to the former one in the sense that any symplectic leaf belongs to a leaf of the anchor foliation.

Remark. A natural question to ask is as follows: Given a Poisson bi-vector (2.20), where ω_{ab} is some non-degenerate 2-form and X_a is an integrable distribution, are these data sufficient to define a symplectic Lie algebroid? In general, the answer is negative, since we do not require the local vector fields X_a to be linearly independent. Nonetheless, if X 's are linearly independent on an everywhere dense domain in M , the answer is positive. In that case the structure equations (2.5) and (2.18) follow immediately from the Jacobi identities for the Schouten commutators of the local vector fields (2.4) and the Poisson bi-vector (2.20). We will discuss this question in more detail in Sect. 5 .

2.3. Symplectic connection and curvature. The deformation quantization of quasi-symplectic Poisson manifolds to be developed in the next sections involves one more geometric ingredient, a *symplectic connection*. This is defined as a torsion-free Lie-algebroid connection respecting a symplectic 2-form, i.e.,

$$(2.22) \quad \nabla_\rho \omega = 0 .$$

Here we consider ω as a section of $\Lambda^0(\mathcal{E}, \wedge^2 \mathcal{E})$.

Proposition 2.2. *Any symplectic Lie algebroid admits a symplectic connection.*

Proof. We are looking for a symplectic connection of the form $\nabla + \Delta\Gamma$, where ∇ is an arbitrary connection and $\Delta\Gamma \in \Gamma(\mathcal{E} \otimes \mathcal{E}^* \otimes \mathcal{E}^*)$. In terms of local coordinates the compatibility condition (2.22) reads

$$(2.23) \quad \nabla_c \omega_{ab} = \Delta\Gamma_{cab} - \Delta\Gamma_{cba} ,$$

where $\Delta\Gamma_{abc} = \Delta\Gamma_{ab}^d \omega_{dc}$. Obviously, these equations cannot have a unique solution: any tensor $\Delta\Gamma'_{abc}$, symmetric in bc , satisfies the homogeneous equation and therefore it can be added to a given solution $\Delta\Gamma_{abc}$ to produce another one. A particular solution to Eq.(2.23) is given by

$$(2.24) \quad \Delta\Gamma_{ab}^c = -\frac{1}{2} \omega^{cd} \nabla_a \omega_{db} ,$$

Now let ∇_ρ be an arbitrary Lie-algebroid connection which respects ω and has torsion T . By making use of the aforementioned ambiguity, one can define the new connection $\nabla'_\rho = \nabla + \Delta\Gamma'$,

$$(2.25) \quad \Delta\Gamma'_{abc} = -\frac{1}{3}(T_{abc} + T_{acb}), \quad T_{abc} = T_{ab}^d \omega_{dc},$$

which is also compatible with ω . By definition (2.12), we have

$$(2.26) \quad T'^c_{ab} = T^c_{ab} + \Delta\Gamma'^c_{ab} - \Delta\Gamma'^c_{ba}.$$

Substituting (2.25) into (2.26) and lowering the upper index with the help of ω we get

$$(2.27) \quad T'_{abc} = T_{acb} + \text{cycle}(a, b, c) = 0.$$

The last equality follows immediately from (2.15) with ω in place of α . Thus, ∇'_ρ is a symplectic connection.

Let R_{abc}^d be the curvature of a symplectic connection. By analogy with Riemannian geometry we can define the *covariant curvature tensor* just lowering upper index with the help of the symplectic 2-form: $R_{abcd} = R_{abc}^n \omega_{nd}$. The following symmetry properties take place:

$$(2.28) \quad R_{abcd} = -R_{bacd}, \quad R_{abcd} = R_{abdc}, \quad R_{abcd} + R_{bcad} + R_{cabd} = 0.$$

The first equality is obvious, the second one follows from the definition (2.8) and the fact that ∇_ρ respects ω , the third equality is just the Bianchi identity (2.16).

2.4. Examples. Let us give some examples of symplectic Lie algebroids and the corresponding quasi-symplectic Poisson brackets. More examples of Lie algebroids, with or without symplectic structure, can be found in [12], [25].

Example 1. Any symplectic manifold (M, ω) gives rise to the symplectic Lie algebroid (TM, id, ω) . The quasi-symplectic Poisson structure is given by

$$(2.29) \quad \alpha = (\omega^{-1})^{\mu\nu} \partial_\mu \wedge \partial_\nu.$$

To get a less trivial quasi-symplectic representation for α consider an almost complex structure J compatible with ω . Recall that an almost complex structure is a smooth field of automorphisms $J : TM \rightarrow TM$ obeying conditions

$$(2.30) \quad J^2 = -\text{id} , \quad \omega(JX, JY) = \omega(X, Y) ,$$

where X, Y are arbitrary vector fields. J being non-degenerate, we get a quasi-symplectic representation for the Poisson bi-vector α associated with the symplectic Lie algebroid (TM, J, ω) :

$$(2.31) \quad \alpha = (\omega^{-1})^{\mu\nu} J_\mu \wedge J_\nu .$$

Here $J = dx^\mu J_\mu^\nu \partial_\nu$ and $J_\mu = J_\mu^\nu \partial_\nu$.

Example 2. Generalizing previous example, consider a pair of Schouten-commuting bi-vectors β and ω , where the former is a Poisson one and the latter is non-degenerate. The triple (T^*M, β, ω) defines a symplectic Lie algebroid with the structure functions

$$[\beta(dx^\mu), \beta(dx^\nu)] = \partial_\lambda \beta^{\mu\nu} \beta(dx^\lambda) , \quad \beta(dx^\mu) = \beta^{\mu\nu} \partial_\nu .$$

The induced Poisson structure on M is given by

$$(2.32) \quad \alpha = (\omega^{-1})_{\mu\nu} \beta^{\mu\gamma} \beta^{\nu\lambda} \partial_\gamma \wedge \partial_\lambda .$$

Example 3. Recall that a Lie algebra L is called *quasi-Frobenius* [28] if it admits a non-degenerate central extension L_c of the form

$$[p_a, p_b] = f_{ab}^d p_d + \omega_{ab} c , \quad [c, p_a] = 0 , \quad \det(\omega_{ab}) \neq 0 .$$

The Jacobi identity for L_c requires the non-degenerate matrix ω_{ab} , determining the central extension, to be a 2-cocycle of the Lie algebra $L \simeq L_c/c$.

Given an action $\rho : L \rightarrow \text{Vect}(M)$ of the Lie algebra L on M by smooth vector fields $X_a = \rho(p_a)$, one can define a symplectic Lie algebroid associated with the trivial vector bundle

$M \oplus L$, anchor ρ , and symplectic form ω_{ab} . The induced quasi-symplectic structure on M reads

$$(2.33) \quad \alpha = \omega^{ab} X_a \wedge X_b, \quad \omega^{ac} \omega_{cb} = \delta_b^a.$$

A simple quantization procedure for such Poisson brackets has been proposed in [15].

Example 4. Let (M, α) be a 2-dimensional Poisson manifold. We say that the bi-vector α is *quasi-homogeneous* if there exist a volume form ω and a vector field Y such that the function $h = \omega(\alpha)$ obeys condition $Yh = h$.

It turns out that any quasi-homogeneous Poisson manifold is also a quasi-symplectic one. Namely, a simple computation yields

$$(2.34) \quad \alpha = X \wedge Y, \quad [X, Y] = (1 - \operatorname{div}_\omega Y)X.$$

Here X is the Hamiltonian vector field associated with the Hamiltonian h and the symplectic (volume) form ω . The structure equations (2.5) and (2.18) are automatically satisfied by the reason of dimension and we get a symplectic Lie algebroid associated with the trivial vector bundle $M \oplus \mathbb{R}^2$.

For instance, the following polynomial Poisson brackets on 2-plane

$$(2.35) \quad \{x, y\} = x^m y^n + x^k y^l,$$

$$\delta = nk - lm \neq 0, \quad m, n, k, l \in \mathbb{N},$$

are quasi-homogeneous w.r.t. $\omega = dx \wedge dy$ and

$$(2.36) \quad \begin{aligned} Y &= \delta^{-1}(n-l)x\partial_x + \delta^{-1}(k-m)y\partial_y, \\ X &= (mx^{m-1}y^n + kx^{k-1}y^l)\partial_y - (nx^m y^{n-1} + lx^k y^{l-1})\partial_x. \end{aligned}$$

In accordance with (2.34)

$$(2.37) \quad [Y, X] = (1 - \delta^{-1}(n+k-l-m))X,$$

and we arrive at the symplectic Lie algebroid associated with the two-dimensional quasi-Frobenius Lie algebra (2.37) (see the previous example).

2.5. Counter-example. As we have seen the quasi-symplectic manifolds constitute a wide class of Poisson manifolds. Here is an example of a Poisson manifold that does not admit any quasi-symplectic representation (even locally).

Proposition 2.3. *The Lie-Poisson bracket on the dual of $so(3)$ algebra does not admit a quasi-symplectic representation.*

Proof. The Poisson bracket in question is of the form

$$(2.38) \quad \{x^i, x^j\} = \sum_k \epsilon^{ijk} x^k,$$

where x^i are linear coordinates in \mathbb{R}^3 and ϵ^{ijk} is the Levi-Civita tensor. The only irregular point is $0 \in \mathbb{R}^3$, where the rank of the Poisson bracket is equal to zero; at the other points the rank equals 2. The leaves of the symplectic foliation $S(\mathbb{R}^3)$ are exactly the level sets of the Casimir function $(x^1)^2 + (x^2)^2 + (x^3)^2$, i.e., spheres centered at the origin.

Since any vector bundle \mathcal{E} over \mathbb{R}^3 is trivial, we may look for an anchor being just an integrable vector distribution $X_a \in \text{Vect}(\mathbb{R}^3)$. For the same reason, any symplectic 2- \mathcal{E} -form is given by an invertible skew-symmetric matrix $\omega_{ab}(x)$ on \mathbb{R}^3 . Clearly, each sphere from $S(\mathbb{R}^3)$ is entirely contained in some leaf of the anchor foliation $F(\mathbb{R}^3)$; so, we write: $S(\mathbb{R}^3) \subset F(\mathbb{R}^3)$. The existence of a quasi-symplectic representation is expressed by the equality

$$(2.39) \quad X_a^i \omega^{ab} X_b^j = \sum_k \epsilon^{ijk} x^k.$$

The key to the analysis of this equation lies with the rank r of the vector distribution at the origin. *A priori*, r may take any value from 0 to 3. Let us show that any assumption about r leads to a contradiction.

The case $r = 0$: This possibility is ruled out by comparing the order of zero on both sides of the equality (2.39). Indeed, since all X 's must vanish at $x = 0$, the order of zero on the l.h.s. is of at least 2, while the r.h.s. tends to zero linearly.

The cases $r = 1, 2$: There is an integral leaf of dimension 1 or 2 passing through the origin and intersecting *transversally* at least one of the symplectic spheres. (Otherwise, this leaf would

be entirely contained in one of the spheres, and thus, could not reach the origin.) But this contradicts to the inclusion $S(\mathbb{R}^3) \subset F(\mathbb{R}^3)$.

The case $r = 3$: Passing, if necessary, to another basis we may assume that $X_a = (X_i, X_\alpha)$, where $X_i = \partial_i$ and $X_\alpha = 0$. Then the matrix ω^{ab} takes the block form

$$(2.40) \quad \begin{pmatrix} \omega^{ij} & \omega^{i\beta} \\ \omega^{\alpha j} & \omega^{\alpha\beta} \end{pmatrix}$$

with $\omega^{ij} = \sum \epsilon^{ijk} x^k$. Among various equations on the matrix elements of (2.40), expressing the fact of closedness of ω , one can find the following one:

$$(2.41) \quad \omega^{\alpha k} \partial_k \omega^{ij} + \omega^{ik} \partial_k \omega^{j\alpha} - \omega^{jk} \partial_k \omega^{i\alpha} = 0.$$

Since $\omega^{ij}(0) = 0$ and $\partial_k \omega^{ij}(0) = \epsilon^{ijk}$, the last equation implies that $\omega^{\alpha i}(0) = 0$, and hence the entire matrix (2.40) must degenerate at the origin. This contradiction concludes the proof.

3. POISSON DESCRIPTION OF SYMPLECTIC LIE ALGEBROIDS

In order to construct as well as physically interpret the deformation quantization of quasi-symplectic manifolds it is convenient to think of (M, α) as the phase space of some (gauge invariant) mechanical system with zero Hamiltonian. In what follows we will use the standard terminology from the theory of constrained systems: first and second class constraints, gauge-fixing conditions, ghost variables, BRST charge etc. [18]. It should be noted that unlike the common practice we will consider Hamiltonian constraints that are defined by a *section* of some vector bundle $\mathcal{E} \rightarrow M$ rather than scalar functions on M . To provide the covariance of the quantization with respect to the bundle automorphisms an appropriate linear connection is needed, and that requires some modification of the conventional BRST formalism [16, 17, 18]. In particular, it will be convenient to use non-canonical commutation relations for ghost variables. The details will be explained below.

Now let us outline the basic steps of our approach. The main idea is to quantize a quasi-symplectic manifold M by means of its suitable embedding into a certain supermanifold endowed with “a more simple” Poisson structure. The construction of such an embedding involves a quite standard machinery of the Hamiltonian BRST theory [17, 18, 29]; it can be subdivided

into three steps. First, using the Lie algebroid structure, we represent (M, α) as a second-class constrained system on the vector bundle \mathcal{E}^* dual to the Lie algebroid \mathcal{E} . As the next step, the second-class constrained system on \mathcal{E}^* is converted into an equivalent gauge system on the direct sum of vector bundles $\mathcal{N} = \mathcal{E}^* \oplus \mathcal{E}$. The equivalence just means that the Poisson algebra of physical observables on \mathcal{N} is isomorphic to the Poisson algebra of smooth functions on (M, α) . Finally, the classical gauge system is covariantly quantized by the BFV-BRST method. The key point is that the space of physical observables on \mathcal{N} , being identified with a certain BFV-BRST cohomology in ghost number zero, carries a simple Poisson structure which can easily be quantized. By construction, the associative product on the algebra of quantum observables on \mathcal{N} induces a $*$ -product on the original quasi-symplectic manifold (M, α) .

For the case of symplectic manifolds, including second-class constrained system, such a program was first implemented in [19], [20] establishing detailed correspondence between the key ingredients of the BRST theory and the Fedosov deformation quantization.

3.1. Symplectic embedding. We start with the description of a symplectic embedding of (M, α) into the dual bundle of the corresponding Lie algebroid. It is well known that \mathcal{E}^* carries a natural Poisson structure, which is dual to the Lie algebroid structure [25], [30]. A proper modification of this Poisson structure in the presence of a symplectic 2-form is offered by the next proposition.

Proposition 3.1. *Let $(\mathcal{E}, \rho, \omega)$ be a symplectic Lie algebroid corresponding to a quasi-symplectic manifold (M, α) . Then $C^\infty(\mathcal{E}^*)$ can be equipped with the following Poisson brackets:*

$$(3.1) \quad \{x^\mu, x^\nu\} = 0, \quad \{p_a, x^\mu\} = X_a^\mu, \quad \{p_a, p_b\} = f_{ab}^c p_c + \omega_{ab}.$$

Here x^μ are local coordinates on M and p_a are linear coordinates on the fibers of \mathcal{E}^* . The Poisson manifold (M, α) is symplectically imbedded into \mathcal{E}^* as zero section.

Remark. Although the definition of the brackets on \mathcal{E}^* involves local coordinates, the Poisson structure (3.1) is actually coordinate independent, so the relationship between the Lie algebroid structure on \mathcal{E} and the Poisson bi-vector on \mathcal{E}^* is intrinsic. The Jacobi identity for (3.1)

generates the full set of the Lie algebroid axioms as well as the closedness condition for the symplectic structure.

In terms of local coordinates (x^μ, p_a) one may identify the base manifold M with those points of \mathcal{E}^* for which

$$(3.2) \quad p_a = 0.$$

Since

$$(3.3) \quad \det(\{p_a, p_b\})|_{p=0} = \det(\omega_{ab}) \neq 0,$$

the canonical imbedding $M \hookrightarrow \mathcal{E}^*$ defined by (3.2) is symplectic, and the induced Poisson structure on M reads

$$(3.4) \quad \{f, g\} = \omega^{ab}(X_a^\mu \partial_\mu f)(X_b^\nu \partial_\nu g), \quad \forall f, g \in C^\infty(M).$$

From the physical viewpoint, this bracket can be thought of as the Dirac bracket associated with the second-class constraints (3.2), where f and g are taken to be p -independent functions on \mathcal{E}^* .

3.2. Classical conversion. Choosing a symplectic connection ∇_ρ , one can extend the Poisson structure (3.1) on \mathcal{E}^* to that on the direct sum $\mathcal{N} = \mathcal{E} \oplus \mathcal{E}^*$. Namely, if y^a are linear coordinates on the fibers of \mathcal{E} , then the corresponding Poisson brackets read

$$(3.5) \quad \begin{aligned} \{x^\mu, x^\nu\} &= 0, & \{p_a, x^\mu\} &= X_a^\mu(x), \\ \{x^\mu, y^a\} &= 0, & \{p_a, y^b\} &= -\Gamma_{ac}^b(x)y^c, \\ \{y^a, y^b\} &= \omega^{ab}(x), & \{p_a, p_b\} &= \omega_{ab}(x) + f_{ab}^c(x)p_c - \frac{1}{2}R_{abcd}(x)y^c y^d. \end{aligned}$$

Here Γ_{ab}^c are the coefficients of the connection ∇_ρ and R_{abcd} is the corresponding curvature tensor.

The brackets (3.5) are well-defined and meet the Jacobi identity. Verifying the Jacobi identity, one gets the compatibility condition (2.22), the definition of the curvature tensor (2.13), the Bianchi identity (2.10), and the axioms of a symplectic Lie algebroid.

Now we aim to replace the second-class constrained system (3.1), (3.2) on \mathcal{E}^* with an equivalent gauge system on the extended Poisson manifold \mathcal{N} . In the Hamiltonian formalism a reparametrization invariant gauge system is completely specified by a set of first class constraints $T_a = 0$ defining some coisotropic submanifold $\Sigma \subset \mathcal{N}$ (a constraint surface). The quotient of Σ by the Hamiltonian action of T 's is assumed to be isomorphic to the quasi-symplectic manifold (M, α) and this is the sense in which the equivalence will be established between the original Poisson manifold and the effective gauge theory. In fact, for the purposes of deformation quantization it is sufficient to work with a *formal gauge system* on \mathcal{N} in the sense that the first class constraints T_a are allowed to be given by formal power series in y 's. It is required, however, that the canonical projection of the formal coisotropic submanifold Σ onto \mathcal{E}^* to coincide with the well-defined constraint surface (3.2), i.e., with M . This allows one to assign a precise meaning to the Hamiltonian reduction by the formal first class constraints.

Thus, we are looking for a set of Hamiltonian constraints $T_a(x, p, y)$ obeying conditions

$$(3.6) \quad \{T_a, T_b\} = U_{ab}^c T_c, \quad T_a(x, p, y)|_{\mathcal{E}} = T_a(x, p, 0) = p_a,$$

where $U_{ab}^c(x, p, y)$ are some structure functions. Geometrically, one can thought of T 's as a section of the vector bundle $\pi : \mathcal{E} \oplus \mathcal{E}^* \rightarrow \mathcal{E}$ over the base \mathcal{E} , with π being the canonical projection onto the first factor.

Proposition 3.2. *The equations (3.6) have a solution of the form*

$$(3.7) \quad T_a = p_a + \sum_{n=1}^{\infty} T_a^n, \quad T_a^n = t_{ab_1 \dots b_n}(x) y^{b_1} \dots y^{b_n},$$

where the coefficients $t_{ab_1 \dots b_n}(x)$ do not depend on p 's.

Remark. In the physical literature, the passage from a given second-class constrained system to an equivalent first-class one is known as the *conversion procedure*; accordingly, y 's are called *conversion variables*. In the local setting, i.e., for a sufficiently small domain in the extended phase space, the existence of the conversion is ensured by a fairly general theorem [29]. Moreover, passing, if necessary, to an equivalent basis of second class constraints, it is possible to

have a solution with $U_{ab}^c = 0$ (abelian conversion). Here, however, we concern with account of global geometry that requires to consider a non-abelian conversion in general.

Proof. Substituting the expansion (3.7) into the involution relations (3.6) and extracting contribution to zero order in y 's, we find

$$(3.8) \quad (f_{ab}^c - U_{ab}^c)p_c + t_{an}\omega^{nm}t_{bm} + \omega_{ab} = 0.$$

A particular solution to this equation is obvious:

$$(3.9) \quad U_{ab}^c = f_{ab}^c, \quad t_{ab} = -\omega_{ab}.$$

Taking this solution, one gets the following chain of equations for higher orders in y 's:

$$(3.10) \quad F_a^s := \partial_{[a}T_{b]}^{s+1} - B_{ab}^s = 0, \quad s \geq 1,$$

where

$$(3.11) \quad \begin{aligned} B_{ab}^1 &= (X_{[a}^i \partial_i \omega_{b]}c + \omega_{d[a} \Gamma_{b]}^d + f_{ab}^d \omega_{dc})y^c, \\ B_{ab}^2 &= \{p_{[a}, T_{b]}^2\} + f_{ab}^c T_c^2 - \frac{1}{2}R_{abcd}y^c y^d, \\ B_{ab}^s &= \{p_{[a}, T_{b]}^s\} + f_{ab}^c T_c^s + \sum_{n=2}^s \{T_a^n, T_b^{s+2-n}\}, \quad s \geq 3. \end{aligned}$$

Hereinafter the square brackets denote anti-symmetrization of indices and ∂_a is the partial derivative with respect to y^a . The form of the equations (3.10) suggests to interpret T_a^s as the components of 1-form $T^s = T_a^s dy^a$ defined on the linear space of y 's. Thus, we can write

$$(3.12) \quad F^s = dT^{s+1} - B^s = 0, \quad s \geq 1,$$

where d is the usual exterior differential with respect to y 's, and B^s is a given 2-form provided the 1-forms T^1, \dots, T^s have been already determined. According to the Poincaré lemma, Eqs.(3.12) are consistent iff the 2-forms B^s are closed. In this case, the general solution to (3.10) reads

$$(3.13) \quad T_a^{s+1} = \frac{1}{s+2}y^b B_{ba}^s + \partial_a C^s,$$

C^s being an arbitrary monomial of degree $s + 2$. The closedness of B 's is now proved by induction in s . Consider the identity

$$(3.14) \quad \{\bar{T}_a^s, \bar{T}_b^s\} - f_{ab}^d \bar{T}_d^s = \sum_{n=0}^{s-1} F_{ab}^n + B_{ab}^s + \cdots,$$

where $\bar{T}_a^s = p_a + \sum_{n=1}^s T_a^n$ and dots stand for the terms of order higher than s . Taking the Poisson bracket of this relation with \bar{T}_c^s and using the Jacobi identity

$$(3.15) \quad \{\{\bar{T}_a^s, \bar{T}_b^s\}, \bar{T}_c^s\} + \text{cycle}(a, b, c) = 0,$$

we can write

$$(3.16) \quad \begin{aligned} & (f_{ab}^d \{\bar{T}_c^s, \bar{T}_d^s\} + \{\bar{T}_c^s, f_{ab}^d \bar{T}_d^s\}) dy^a \wedge dy^b \wedge dy^c = \\ & = \left(\sum_{n=0}^{s-1} \{F_{ab}^n, \bar{T}_c^s\} - \{B_{ab}^s, \bar{T}_c^s\} + \cdots \right) dy^a \wedge dy^b \wedge dy^c. \end{aligned}$$

With account of (3.5) and the Lie algebroid relations (2.5), the contribution to the $(s - 1)$ -th order of the last equation is given by

$$(3.17) \quad dB^s = \left(f_{ab}^d F_{dc}^{s-1} + \sum_{n=0}^{s-1} \{F_{ab}^n, T_c^{s-n+2}\} \right) dy^a \wedge dy^b \wedge dy^c.$$

But the r.h.s. of this relation vanishes by the induction hypothesis. Thus, B^s is a closed 2-form and the recurrent formula (3.13) gives the general solution for T_a .

Notice that the ambiguity concerning the choice of arbitrary functions C^s , entering the general solution for T_a , can be removed by imposing the y -transversality condition

$$(3.18) \quad y^a T_a^s = 0, \quad s \geq 1.$$

Then it follows from Eq. (3.13) that

$$y^a \partial_a C^s(x, y) = (s + 2) C^s(x, y) = 0 \Rightarrow C^s(x, y) = 0.$$

Remark. For the case of symplectic manifolds, the convergence of the series (3.7) in a tubular neighborhood of M was proved in [31] under assumptions of analyticity and compactness. It seems that the same arguments are applicable to any quasi-symplectic manifold provided all the structure functions are real-analytical and M is compact.

Now to see the equivalence of the constructed gauge system on \mathcal{N} to the original Hamiltonian system on M it suffices to note that equations $\chi^a := y^a = 0$ are well-defined gauge-fixing conditions for the first class constraints $T_a = 0$. Indeed,

$$(3.19) \quad \det \left(\begin{array}{cc} \{T_a, T_c\} & \{T_a, \chi^d\} \\ \{\chi^b, T_c\} & \{\chi^b, \chi^d\} \end{array} \right) \Big|_{T=\chi=0} = \det \left(\begin{array}{cc} 0 & -\delta_a^d \\ \delta_c^b & \omega^{bd} \end{array} \right) = 1.$$

Therefore, the reduced Poisson manifold (physical phase space) is isomorphic to the constraint surface $T_a = \chi^b = 0$. The last equations are obviously equivalent to $p_a = y^a = 0$, i.e., defines the canonical projection $\varphi : \mathcal{N} \rightarrow M$. The explicit description of the resulting Poisson structure on M can be obtained by introducing the Dirac bracket for the second-class constraints (T_a, χ^b) . Identifying the space of smooth functions on M with the subspace of p - and y -independent functions on \mathcal{N} , it is easy to see that $\varphi : \mathcal{N} \rightarrow M$ is a Poisson map relating the Dirac bracket $\{\cdot, \cdot\}_D$ on \mathcal{N} with the initial quasi-symplectic bracket on M , i.e.,

$$(3.20) \quad \varphi_*(\{f, g\}_D) = \{\varphi_*(f), \varphi_*(g)\}_M.$$

4. QUANTIZATION

Having realized the quasi-symplectic manifold (M, α) as a formal gauge system on \mathcal{N} we are ready to perform its BRST quantization. As usual, this implies further enlargement of the phase space of the system by ghost variables, constructing a nilpotent BRST charge, and identifying physical observables with certain BRST cohomology classes.

4.1. Ghost variables and the classical BRST charge. With each first class constraint T_a we associate the pair of anticommuting (Grassman odd) ghost variables $(\mathcal{C}^a, \mathcal{P}_b)$ subject to the canonical Poisson bracket relations

$$(4.1) \quad \{\mathcal{C}^a, \mathcal{P}_b\} = \delta_b^a, \quad \{\mathcal{C}^a, \mathcal{C}^b\} = \{\mathcal{P}_a, \mathcal{P}_b\} = 0.$$

It is quite natural to treat \mathcal{C}^a and \mathcal{P}_b as linear coordinates on the fibers of the vector bundles $\Pi\mathcal{E}$ and $\Pi\mathcal{E}^*$, respectively. Here by Π we denote the parity reversion operation: being applied to a vector bundle it transforms the bundle into the super-vector bundle with the same base manifold and transition functions, and the fibers being the Grassman odd vector spaces. Thus,

the phase space of our gauge system is extended to the direct sum of (super-)vector bundles $\mathcal{M} = \mathcal{N} \oplus \Pi\mathcal{N}$. This geometric interpretation places the ghosts on equal footing with the conversion variables y 's and suggests the following extension of the Poisson structure from \mathcal{N} to \mathcal{M} :

$$(4.2) \quad \{p_a, \mathcal{C}^b\} = -\Gamma_{ac}^b(x)\mathcal{C}^c, \quad \{p_a, \mathcal{P}_b\} = \Gamma_{ab}^c(x)\mathcal{P}_c.$$

The brackets of the ghosts with x^μ and y^a are equal to zero. To meet the Jacobi identity one has to modify the Poisson brackets of p 's by ghost terms as follows

$$(4.3) \quad \{p_a, p_b\} = \omega_{ab}(x) + f_{ab}^c(x)p_c - \frac{1}{2}R_{abcd}(x)y^c y^d - R_{abc}^d(x)\mathcal{C}^c \mathcal{P}_d.$$

The other Poisson brackets (3.5) remain intact.

Remark. At this point we slightly deviate from the usual line of the BRST scheme, where the ghost variables are assumed to Poisson-commute with functions on the original phase space (\mathcal{N} in our case) and, in particular, with the first class constraints. In principle, it is possible to work with the canonical Poisson brackets for ghosts, setting the r.h.s of (4.2) to zero and omitting the last term in (4.3), but this leads to nonlinear transformations of p_a under bundle automorphisms (p 's are mixed with the ghost bilinears $\mathcal{C}^a \mathcal{P}_b$). We refer to [19] for the details of this construction in the case where M is a symplectic manifold (2.29). As we will see bellow, these non-canonical Poisson brackets of ghosts can be naturally incorporated into the BRST quantization procedure making it explicitly covariant.

Let $\mathcal{F}(M)$ denote the super-Poisson algebra of functions on the supermanifold \mathcal{M} ; the elements of $\mathcal{F}(\mathcal{M})$ are superfunctions of the form²

$$(4.4) \quad A(x, p, y, \mathcal{C}, \mathcal{P}) = \sum A_{a_1 \dots a_k d_1 \dots d_n}^{b_1 \dots b_m}(x) y^{a_1} \dots y^{a_k} \mathcal{C}^{d_1} \dots \mathcal{C}^{d_n} \mathcal{P}_{b_1} \dots \mathcal{P}_{b_m},$$

where $A_{a_1 \dots a_k d_1 \dots d_n}^{b_1 \dots b_m}(x)$ are \mathcal{E} -tensors. In addition to the usual \mathbb{Z}_2 -grading, associated with the Grassman parity,

$$(4.5) \quad \epsilon(\mathcal{C}^a) = \epsilon(\mathcal{P}_b) = 1, \quad \epsilon(x^i) = \epsilon(p_a) = \epsilon(y^b) = 0 \pmod{2},$$

²In what follows we omit the prefix “super” whenever possible.

the space $\mathcal{F}(\mathcal{M})$ is endowed with an additional \mathbb{Z} -grading by prescribing the following *ghost numbers* to the local coordinates:

$$(4.6) \quad \text{gh}(\mathcal{C}^a) = 1, \quad \text{gh}(\mathcal{P}_a) = -1, \quad \text{gh}(x^i) = \text{gh}(p_a) = \text{gh}(y^a) = 0.$$

The ghost number just counts the difference between powers of \mathcal{C} 's and \mathcal{P} 's, entering homogeneous elements of $\mathcal{F}(\mathcal{M})$, and is additive with respect to the Poisson algebra operations:

$$(4.7) \quad \text{gh}(AB) = \text{gh}(A) + \text{gh}(B), \quad \text{gh}(\{A, B\}) = \text{gh}(A) + \text{gh}(B).$$

In particular, functions with zero ghost number form a subalgebra in the Poisson algebra $\mathcal{F}(\mathcal{M})$.

The classical BRST charge $Q \in \mathcal{F}(\mathcal{M})$ is defined as an odd function of ghost number 1 obeying the *classical master equation*

$$(4.8) \quad \{Q, Q\} = 0,$$

and the standard boundary conditions

$$(4.9) \quad Q|_{\mathcal{P}=0} = \mathcal{C}^a T_a.$$

A function $a \in \mathcal{F}(\mathcal{M})$ is said to be BRST invariant if

$$(4.10) \quad DA := \{Q, A\} = 0, \quad \text{gh}(A) = 0.$$

Clearly, $D^2 = 0$. The space of *physical observables* is identified with the zero-ghost-number cohomology of the BRST operator D . The Poisson algebra structure on $\mathcal{F}(\mathcal{M})$ induces that on the space of physical observables.

According to general theorems of the BRST theory [18], (i) Eq. (4.8) is always soluble, and (ii) the Poisson algebra of physical observables is isomorphic to that obtained by the Hamiltonian reduction by the first class constraints. In the case at hands, these statements can be refined as follows.

Proposition 4.1. *The classical master equation (4.8) admits the following solution:*

$$(4.11) \quad Q = \mathcal{C}^a T_a.$$

The Poisson algebra of physical observables on \mathcal{M} is isomorphic to that on the quasi-symplectic manifold (M, α) . Each physical observable can be represented by a BRST invariant element from $\mathcal{F}(\mathcal{M})$ that does not depend on the ghost variables.

Proof. The first part of the proposition is easily verified by straightforward calculations. Notice that, unlike what one has in the standard BRST theory, the first class constraints T_a are no longer in involution as we have modified the Poisson brackets of p 's by the ghost-dependent term (4.3). Luckily, this term does not contribute to the nilpotency condition due to the symmetry properties of the curvature tensor (2.28).

The rest of the proposition will follow from the classical limit of the analogous statement for the quantum BRST observables to be considered in the next section. Here we just show that each physical observable $A(x, p, y, \mathcal{C}, \mathcal{P})$ on \mathcal{M} is uniquely determined by its projection $A(x, 0, 0, 0, 0)$ on M . Speaking informally, this implies that the space of physical observables is not larger than $C^\infty(M)$. In order to see this, let us introduce the following *homotopy operator* $h : \mathcal{F}(\mathcal{M}) \rightarrow \mathcal{F}(\mathcal{M})$:

$$(4.12) \quad h = \mathcal{P}_a \frac{\partial}{\partial \bar{p}_a} + y^a \frac{\partial}{\partial \mathcal{C}^a},$$

where $\bar{p}_a := p_a - \omega_{ab} y^b$. From the explicit expression for the BRST charge (4.11) it follows that

$$(4.13) \quad D = \bar{p}_a \frac{\partial}{\partial \mathcal{P}_a} + \mathcal{C}^a \frac{\partial}{\partial y^a} + \cdots,$$

Here the dots stand for the terms that increase the total degree when acting on monomials in $y, \mathcal{C}, \mathcal{P}$ and p . Then

$$(4.14) \quad Dh + hD = N,$$

where $N = N_0 + \cdots$, and

$$(4.15) \quad N_0 = y^a \frac{\partial}{\partial y^a} + \bar{p}_a \frac{\partial}{\partial \bar{p}_a} + \mathcal{C}^a \frac{\partial}{\partial \mathcal{C}^a} + \mathcal{P}_a \frac{\partial}{\partial \mathcal{P}_a}.$$

Obviously, $\ker N_0 = C^\infty(M) \subset \mathcal{F}(\mathcal{M})$, and hence the operator N_0 is invertible on the subspace $\mathcal{F}_0 = \mathcal{F}(\mathcal{M}) \setminus C^\infty(M)$ and so is the operator N . This implies that the BRST cohomology is

centered in the subspace $C^\infty(M)$; for any BRST invariant B from the complementary subspace \mathcal{F}_0 we have

$$(4.16) \quad B = DC, \quad C = h(N|_{\mathcal{F}_0})^{-1}B.$$

To conclude this section, let us depict the diagram of maps describing the path from the original quasi-symplectic manifold (M, α) to the super-Poisson manifold \mathcal{M} :

$$(4.17) \quad \mathcal{M} = \mathcal{N} \oplus \Pi\mathcal{N} \rightarrow \mathcal{N} = \mathcal{E} \oplus \mathcal{E}^* \rightarrow \mathcal{E} \rightarrow M.$$

All the arrows are canonical projections.

4.2. Quantization of the super-Poisson manifold \mathcal{M} . In the general case, it is not easy to quantize the irregular Poisson brackets (3.5), (4.1) - (4.3). Fortunately, for our purposes, it is sufficient to deal with a special Poisson subalgebra $\mathcal{A} \subset \mathcal{F}(\mathcal{M})$. This is given by functions of $x^\mu, y^a, \mathcal{C}^a$ and $p := \mathcal{C}^a p_a$. Since $p^2 = 0$, the generic element of \mathcal{A} has the form

$$(4.18) \quad A(x, y, p, \mathcal{C}) = a(x, y, \mathcal{C}) + b(x, y, \mathcal{C})p,$$

where a, b belong to the Poisson subalgebra \mathcal{A}_0 of p -independent elements of \mathcal{A} . The elements of \mathcal{A}_0 are given thus by formal power series in y 's and \mathcal{C} 's with coefficients in $\Lambda(\mathcal{E}, S(\mathcal{E}))$, $S(\mathcal{E})$ being the space of symmetric tensor powers of \mathcal{E} . With this geometrical interpretation the basic Poisson bracket relations in \mathcal{A} can be written as

$$(4.19) \quad \{p, p\} = R + \omega, \quad \{p, a\} = \nabla a, \quad \{a, b\} = \omega^{cd} \frac{\partial a}{\partial y^c} \frac{\partial b}{\partial y^d}, \quad a, b \in \mathcal{A}_0.$$

Here

$$(4.20) \quad \nabla a = \mathcal{C}^d \nabla_d a = \mathcal{C}^a \left(X_a^\mu \frac{\partial}{\partial x^\mu} - y^b \Gamma_{ab}^d \frac{\partial}{\partial y^d} - \mathcal{C}^b \Gamma_{ab}^d \frac{\partial}{\partial \mathcal{C}^d} \right) a$$

is the covariant derivative in \mathcal{A}_0 induced by the symplectic connection ∇_ρ on $\Lambda(\mathcal{E}, S(\mathcal{E}))$, and

$$(4.21) \quad R = -\frac{1}{2} R_{abcd} \mathcal{C}^a \mathcal{C}^b y^c y^d, \quad \omega = \omega_{ab} \mathcal{C}^a \mathcal{C}^b, \quad R, \omega \in \mathcal{A}_0,$$

are the covariant curvature tensor and the symplectic form written in the frame (y^a, \mathcal{C}^a) .

In view of Proposition 4.1, the algebra \mathcal{A} contains the classical BRST charge (5.23) as well as all the physical observables of the effective gauge system. It is the reason why one can

restrict consideration to the subalgebra \mathcal{A} when the goal is to quantize the algebra of physical observables.

Proceeding to quantization, we introduce the formal deformation parameter \hbar and extend the Poisson algebra \mathcal{A} to the tensor product

$$(4.22) \quad \hat{\mathcal{A}} = \mathcal{A} \otimes [[\hbar]] ,$$

where $[[\hbar]]$ denotes the space of formal power series in \hbar with coefficients in \mathbb{C} . Accordingly, denote by $\hat{\mathcal{A}}_0 := \mathcal{A}_0 \otimes [[\hbar]]$ the subalgebra of p -independent elements of $\hat{\mathcal{A}}$. There is an almost obvious quantum product giving rise to deformation quantization of the Poisson algebra $\hat{\mathcal{A}}$. For any two elements $a, b \in \hat{\mathcal{A}}_0$ we just use the Weyl-Moyal formula

$$(4.23) \quad (a \circ b)(x, y, \mathcal{C}, \hbar) = \exp \left(-\frac{i\hbar}{2} \omega^{ab} \frac{\partial}{\partial y^a} \frac{\partial}{\partial z^b} \right) a(x, y, \mathcal{C}, \hbar) b(x, z, \mathcal{C}, \hbar) \Big|_{y=z} .$$

and then extend this \circ -product to the whole algebra $\hat{\mathcal{A}}$ by associativity setting

$$(4.24) \quad p \circ a = pa - i\hbar \nabla a, \quad a \circ p = ap, \quad p \circ p = -i\hbar(R + \omega) .$$

Clearly, the \circ -product respects both the Grassman and the ghost-number gradings.

As for any graded associative algebra, we can endow $\hat{\mathcal{A}}$ with the structure of super-Lie algebra w.r.t. the super-commutator

$$(4.25) \quad [A, B] = A \circ B - (-1)^{\epsilon(A)\epsilon(B)} B \circ A ,$$

A, B being homogeneous elements of $\hat{\mathcal{A}}$.

For further purposes let us introduce one more useful grading on $\hat{\mathcal{A}}$ by prescribing the following degrees to the variables:

$$(4.26) \quad \deg(x^\mu) = \deg(\mathcal{C}^a) = 0, \quad \deg(y^a) = 1, \quad \deg(p) = \deg(\hbar) = 2 .$$

Since this grading involves essentially the deformation parameter we will refer to it as \hbar -grading.

4.3. Quantum BRST charge. This is defined as an element $\hat{Q} \in \hat{\mathcal{A}}$ of ghost number 1 satisfying the *quantum master equation*

$$(4.27) \quad [\hat{Q}, \hat{Q}] = 2\hat{Q} \circ \hat{Q} = 0$$

with the boundary condition

$$(4.28) \quad \hat{Q}|_{\mathcal{P}=\hbar=0} = \mathcal{C}^a T_a.$$

The adjoint action of \hat{Q} defines the nilpotent derivation $\hat{D} : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$:

$$(4.29) \quad \hat{D}a = \frac{i}{\hbar} [\hat{Q}, a], \quad a \in \hat{\mathcal{A}}.$$

The operator \hat{D} increases the ghost number by 1 preserving the subalgebra $\hat{\mathcal{A}}_0$.

By definition, the space of *quantum physical observables* is identified with the zero-ghost-number cohomology of the operator \hat{D} .

Let us show the existence of a quantum BRST charge \hat{Q} whose classical limit coincides with the classical BRST charge Q . Technically, instead of finding \hbar -corrections to Q , it is more convenient to build up \hat{Q} using recursion on the total \hbar -degree (4.26). In order to do this we introduce the pair of Fedosov's operators changing the \hbar -degree by 1 unit. The first operator is given by

$$(4.30) \quad \delta a = \mathcal{C}^a \frac{\partial a}{\partial y^a}, \quad \delta^2 = 0.$$

for any $a \in \hat{\mathcal{A}}_0$. Since

$$(4.31) \quad \delta a = \frac{i}{\hbar} [\mathcal{C}^a \omega_{ab} y^b, a],$$

it is an internal derivation of $\hat{\mathcal{A}}_0$. The second operator is defined by its action on homogeneous functions:

$$(4.32) \quad \begin{aligned} \delta^* a_{mn} &= \frac{1}{n+m} y^a \frac{\partial a}{\partial \mathcal{C}^a}, \quad n+m \neq 0, \\ \delta^* a_{00} &= 0, \end{aligned}$$

where $a_{nm} = a_{a_1 \dots a_n b_1 \dots b_m}(x, \hbar) y^{a_1} \dots y^{a_n} \mathcal{C}^{b_1} \dots \mathcal{C}^{b_m}$. Like δ , the operator δ^* is nilpotent, though it is not a derivation of the \circ -product. One can regard δ^* as a homotopy operator for δ :

$$(4.33) \quad a|_{\mathcal{C}=y=0} + \delta\delta^*a + \delta^*\delta a = a, \quad \forall a \in \hat{\mathcal{A}}_0.$$

The last relation resembles the usual Hodge-De Rham decomposition for the exterior algebra of differential forms.

Proposition 4.2. *The quantum master equation (4.27) has a solution of the form*

$$(4.34) \quad \hat{Q} = \sum_{r=1}^{\infty} Q_r, \quad \deg(Q_r) = r,$$

where

$$(4.35) \quad Q_1 = -\mathcal{C}^a \omega_{ab} y^b, \quad Q_2 = p, \quad \text{and} \quad Q_r \in \hat{\mathcal{A}}_0, \quad \forall r > 2,$$

which is unique if we require

$$(4.36) \quad \delta^* Q_r = 0, \quad \forall r > 2.$$

Proof. The first three terms in the expansion (4.34) coincide with those in the classical BRST charge (4.11), and this proves the validity of Eq. (4.27) in the lowest order in \hbar -degree. For $r \geq 4$ Eq. (4.27) implies

$$(4.37) \quad \delta Q_{r+1} = B_r,$$

where

$$(4.38) \quad B_r = -\frac{i}{2\hbar} \sum_{s=0}^{r-2} [Q_{r+2}, Q_{r-s}], \quad \deg(B_r) = r.$$

In view of the Hodge-De Rham decomposition (4.33), Eq. (4.37) is soluble iff $\delta B_r = 0$. In this case we have the unique solution

$$(4.39) \quad Q_{r+1} = \delta^* B_r$$

subject to the extra condition $\delta^* Q_{r+1} = 0$. So, it remains to show that $\delta B_r = 0$. Proceeding by induction, we assume that Eq. (4.27) is valid up to the s -th order in \hbar -degree. Then, extracting the $(s+3)$ -order in the Jacobi identity

$$(4.40) \quad [Q^s, [Q^s, Q^s]] = 0, \quad Q^s = \sum_{k=1}^s Q_k,$$

one gets $\delta B_s = 0$, that completes the proof.

Since $\text{gh}(Q) = 1$, the ansatz (4.34) implies the following structure of the quantum BRST charge: $Q = \mathcal{C}^a \hat{T}_a(x, y, \hbar)$, where \hat{T}_a are the “quantum” first class constraints. Then Rel. (4.36) is just another form of the y -transversality condition (3.18), which allows one to extract a unique solution both at the classical and quantum levels.

4.4. Quantum observables and star-product. In Sect. 4.1. we have shown that the space of physical observables of the classical gauge system on \mathcal{N} is not larger than $C^\infty(M)$ in the sense that any physical observable is uniquely determined by its projection on M . In this section we prove the inverse: any physical observable on M , has a unique BRST-invariant extension to a zero-ghost-number function from \mathcal{A}_0 . Moreover, this picture takes place at the quantum level as well if we replace $C^\infty(M) \rightarrow C^\infty(M) \otimes [[\hbar]]$ and $\mathcal{A}_0 \rightarrow \hat{\mathcal{A}}_0$. Therefore, it is sufficient to consider only the quantum case, the classical statement will follow from the classical limit.

Proposition 4.3. *For any $a \in C^\infty(M)$ there is a unique $\hat{a} \in \hat{\mathcal{A}}_0$ obeying conditions*

$$(4.41) \quad [\hat{Q}, \hat{a}] = 0, \quad \hat{a}|_{y=0} = a.$$

Proof. Consider the expansion of $\hat{a} \in \hat{\mathcal{A}}_0$ according to the \hbar -degree:

$$(4.42) \quad \hat{a} = \sum_{s=0}^{\infty} a_s, \quad \deg(a_s) = s.$$

The second condition in (4.41) says that $a_0 = a(x)$. Substituting this expansion into the first equation, one gets

$$(4.43) \quad \delta a_{s+1} = B_s, \quad s > 1,$$

where

$$(4.44) \quad B^s = \frac{i}{\hbar} \sum_{k=0}^{s-2} [Q^{k+2}, a^{s-k}], \quad \deg(B^s) = s.$$

In view of the Hodge-De Rham decomposition (4.33) and the boundary condition (4.41), Eq. (4.43) has the unique solution

$$(4.45) \quad a^{s+1} = \delta^* B^s,$$

provided the r.h.s. is δ -closed. The equality $\delta B^s = 0$ follows by induction from the $(s+3)$ -order of the identity

$$(4.46) \quad [\hat{Q}, [\hat{Q}, \hat{a}]] = 0.$$

Corollary. *There is a linear isomorphism between the spaces of quantum observables on M , i.e., $C^\infty(M) \otimes [[\hbar]]$, and the zero-ghost-number cohomology of the BRST-differential $\hat{D} : \hat{\mathcal{A}}_0 \rightarrow \hat{\mathcal{A}}_0$.*

Proof. Eq. (4.41), being linear, has a unique solution even though we allow $\hat{a}|_{y=0}$ to depend formally on \hbar . Therefore, we can replace $C^\infty(M)$ with $C^\infty(M) \otimes [[\hbar]]$.

Clearly, the \circ -product on $\hat{\mathcal{A}}_0$ descends to the BRST cohomology and, in view of the corollary, induces an associative $*$ -product on $C^\infty(M) \otimes [[\hbar]]$. Explicitly,

$$(4.47) \quad a * b = (\hat{a} \circ \hat{b})|_{y=0} = \sum_{n=1}^{\infty} \hbar^n D_n(a, b), \quad \forall a, b \in C^\infty(M) \otimes [[\hbar]],$$

where

$$(4.48) \quad D_0(a, b) = ab, \quad D_1(a, b) = -\frac{i}{2} \{a, b\}_M,$$

and the “hat” stands for the BRST-invariant lift from M to \mathcal{M} (the existence and uniqueness of such a lift are ensured by Proposition 4.3). The higher orders in \hbar , being recurrently constructed by (4.39) and (4.45), involve also the symplectic connection and the curvature.

Remark. By construction, the bi-differential operators D_n entering the $*$ -product (4.47) have a rather peculiar structure. Namely, they are determined by repeated differentiations along the

anchor distribution $\{X_a\}$:

$$(4.49) \quad D_n(a, b) = \sum_{k, l \leq n} D_n^{c_1 \dots c_k d_1 \dots d_l}(x) (X_{c_1} \dots X_{c_k} a) (X_{d_1} \dots X_{d_l} b).$$

Here the structure functions $D^{c_1 \dots c_k d_1 \dots d_l}(x)$ are *universally* expressed via the data of a symplectic Lie algebroid and a Lie algebroid connection. The differential operators of the form (4.49) are called the \mathcal{E} -differential operators; accordingly, the $*$ -product (4.47) is called the \mathcal{E} -deformation of M . As was shown in [11], any \mathcal{E} -deformation of M can be induced by an \mathcal{E} -deformation of the commutative algebra of \mathcal{E} -jets. Conversely, the \mathcal{E} -deformation of M , given by the formula (4.47), admits a canonical extension to the space of \mathcal{E} -jets (by universality). In [15] such an extension was used to derive the universal deformation formula for triangular Lie bialgebras.

5. FACTORIZABLE POISSON BRACKETS BEYOND SYMPLECTIC LIE ALGEBROIDS.

As we have seen, the concept of a symplectic Lie algebroid gives rise to an interesting class of Poisson brackets. Not any Poisson bracket comes in this way, but when it does, we have a simple quantization procedure generalizing the Fedosov quantization. In this section we would like to discuss, in a sense, an inverse problem: To what extent the factorization (2.20) of a Poisson bi-vector α defines a symplectic Lie algebroid?

The precise formulation of the problem is as follows. Let $\mathcal{E} \rightarrow M$ be a vector bundle over a smooth manifold M , ω a section of $\mathcal{E} \wedge \mathcal{E}$, and X a section of $\mathcal{E}^* \otimes TM$. By a slight abuse of notation, we will use the same letters ω and X to denote the corresponding bundle homomorphisms $\omega : \mathcal{E}^* \rightarrow \mathcal{E}$ and $X : \mathcal{E} \rightarrow TM$. Let us also suppose that the \mathcal{E} -bi-vector ω is non-degenerate (defines an isomorphism between \mathcal{E} and its dual \mathcal{E}^*) and X is involutive. The latter means that in each trivializing coordinate chart $\mathcal{U} \subset M$ with frame $s_\alpha \in \Gamma(\mathcal{E}|_{\mathcal{U}})$, the local vector fields $X_\alpha = X_\alpha^i \partial_i \in \text{Vect}(\mathcal{U})$ form an involutive distribution,

$$(5.1) \quad [X_\alpha, X_\beta] = f_{\alpha\beta}^\gamma X_\gamma,$$

$f_{\alpha\beta}^\gamma$ being smooth functions on \mathcal{U} . Clearly, the property of $\{X_\alpha\}$ to be involutive does not depend on a frame, and hence, $\{X_\alpha\}$ generates a (singular) foliation $F(M)$. Suppose now that

the bi-vector

$$(5.2) \quad \alpha = \omega^{\alpha\beta} X_\alpha \wedge X_\beta \in \wedge^2 TM$$

satisfies the Jacobi identity

$$(5.3) \quad [\alpha, \alpha] = 0.$$

Question: *What is the most general geometric structure underlying Eqs. (5.1-5.3).*

A particular solution to these equations is delivered by a symplectic Lie algebroid $\mathcal{E} \rightarrow M$ with anchor X and symplectic 2-form ω . In this case (M, α) is just a quasi-symplectic manifold considered in the previous sections.

Explicitly, the Jacobi identities for the local vector fields X_a and the Poisson bi-vector α read

$$(5.4) \quad (f_{\alpha\beta}^\delta f_{\delta\gamma}^\mu - X_\gamma f_{\alpha\beta}^\mu + \text{cycle}(\alpha, \beta, \gamma)) X_\mu = 0,$$

$$(5.5) \quad (\omega_{\gamma\delta} f_{\alpha\beta}^\delta - X_\gamma \omega_{\alpha\beta}) X^\alpha \wedge X^\beta \wedge X^\gamma = 0,$$

$$X^\alpha = \omega^{\alpha\beta} X_\beta, \quad \omega_{\alpha\gamma} \omega^{\gamma\beta} = \delta_\alpha^\beta.$$

If the map $X : \mathcal{E} \rightarrow TM$ is injective on an everywhere dense domain in M , the expressions in parentheses (5.5) must vanish by continuity, and we arrive at a symplectic Lie algebroid (\mathcal{E}, X, ω) . In the opposite case the l.h.s. of Eqs. (5.4), (5.5) cannot be “divided” by X_a so simply, and thus, more general solutions for the structure functions $f_{\alpha\beta}^\gamma$, $\omega_{\alpha\beta}$ and X_α^i are possible. To further study these equations, we impose a certain regularity condition on X . In what follows we will assume that the space $\Gamma(TM)$, considered as a $C^\infty(M)$ -module, admits a resolution of the form

$$(5.6) \quad 0 \leftarrow TM \xleftarrow{d_1} \mathcal{E}_1 \xleftarrow{d_2} \mathcal{E}_2 \leftarrow \dots \leftarrow \mathcal{E}_{n-1} \xleftarrow{d_n} \mathcal{E}_n \leftarrow 0,$$

where $\mathcal{E}_k \rightarrow M$ and d_k are sequences of vector bundles over M and their M -morphisms with $\mathcal{E}_1 = \mathcal{E}$ and $d_1 = X$. (A sequence of homomorphisms of modules (5.6) is called a resolution of the module $\Gamma(TM)$, if $\text{im } d_{k+1} = \ker d_k$. In other words, the sequence (5.6) is just a cochain complex,

which is exact, exclusive of maybe the first term.) Here we do not require the morphisms d_k to have constant ranks, but since $n < \infty$, their ranks have to be constant on an open everywhere dense domain in M . In particular, the last structure map $d_k : \mathcal{E}_k \rightarrow \mathcal{E}_{k-1}$ should be injective on an everywhere dense domain in M . By analogy with ordinary Lie algebroids, we will refer to the first structure map $d_1 = X$ as anchor.

In order to clarify the meaning of the regularity condition, let us choose an open domain $\mathcal{U} \subset M$ such that for all $k = 1, \dots, n$, $\mathcal{E}_k|_{\mathcal{U}}$ is a trivial vector bundle with frame s_{α_k} . Upon restriction on \mathcal{U} , the morphisms d_k are represented by matrices $d_{\alpha_k}^{\alpha_{k-1}}$, so that $d_{\alpha_k}^{\alpha_{k-1}} d_{\alpha_{k+1}}^{\alpha_k} = 0$. Since the complex (5.6) is exact starting with d_1 , the equality $f^{\alpha_k} d_{\alpha_k}^{\alpha_{k-1}} = 0$, f^{α_k} being a section of \mathcal{E}_k , implies that $f^{\alpha_k} = g^{\alpha_{k+1}} d_{\alpha_{k+1}}^{\alpha_k}$ for some section $g^{\alpha_{k+1}}$ of \mathcal{E}_{k+1} .

Example 0. Consider the adjoint representation of $so(3)$. Identifying the carrier space $so(3)$ with \mathbb{R}^3 we get a set of three linear vector fields on \mathbb{R}^3 generating the $so(3)$ -algebra action:

$$(5.7) \quad \text{ad}_i = \epsilon_{ijk} x_j \partial_k, \quad [\text{ad}_i, \text{ad}_j] = -\epsilon_{ijk} \text{ad}_k.$$

Clearly, the rank of the anchor $\text{ad} : \mathbb{R}^3 \times so(3) \rightarrow T\mathbb{R}^3$ equals 2 in general position and vanishes at the origin $0 \in \mathbb{R}^3$. Since the equation $f^i(x) \text{ad}_i = 0$ implies $f^i(x) = g(x) x^i$, for some smooth function g , while the equation $x^i h(x) = 0$ has the unique solution $h = 0$, we get the following resolution:

$$(5.8) \quad 0 \leftarrow T\mathbb{R}^3 \xleftarrow{\text{ad}} \mathbb{R}^3 \times so(3) \xleftarrow{d_2} \mathcal{E}_2 \leftarrow 0,$$

where \mathcal{E}_2 is a linear bundle over \mathbb{R}^3 and $d_2 = (x^i)$.

Given an anchor X satisfying the regularity condition, one can solve the Jacobi identity (5.4) in the following form:

$$(5.9) \quad f_{\alpha\beta}^{\delta} f_{\delta\gamma}^{\mu} - X_{\gamma} f_{\alpha\beta}^{\mu} + \text{cycle}(\alpha, \beta, \gamma) = f_{\alpha\beta\gamma}^a d_a^{\mu},$$

where $f_{\alpha\beta\gamma}^a$ are smooth functions on \mathcal{U} , skew-symmetric in $\alpha\beta\gamma$, and d_a^{α} is the matrix of the second structure map d_2 in (5.6). By definition, we have

$$(5.10) \quad d_a^{\gamma} X_{\gamma} = 0.$$

In order to solve the Jacobi identity for α , we assume the anchor foliation $F(M)$ to be regular³, i.e., $\text{im}X$ is an integrable subbundle of TM . Then

$$(5.11) \quad \omega_{\gamma\delta}f_{\alpha\beta}^\delta - X_\gamma\omega_{\alpha\beta} + \text{cycle}(\alpha, \beta, \gamma) = W_{\alpha\beta}^a d_{a\gamma} + \text{cycle}(\alpha, \beta, \gamma), \quad d_{a\gamma} = \omega_{\gamma\beta}d_a^\beta,$$

$W_{\alpha\beta}^a = -W_{\beta\alpha}^a$ being smooth functions on \mathcal{U} . Examining compatibility of these equations with the involution relations (5.1), one obtains an infinite set of higher structure functions and structure relations to be studied below.

Let us forget for a moment about the Poisson bi-vector α , focusing at the anchor distribution X . Commuting (5.10) with X_β , we find

$$(5.12) \quad X_\beta d_a^\alpha + f_{\beta\gamma}^\alpha d_a^\gamma = -f_{\beta a}^b d_b^\alpha,$$

f_{ba}^β being local functions. Contracting the last identity with d_c^β and symmetrizing in the indices ac , we get

$$(5.13) \quad d_c^\beta f_{\beta a}^b + d_a^\beta f_{\beta c}^b = f_{ca}^A d_A^b,$$

where d_A^b is the matrix of the third structure map in (5.6), so that $d_A^b d_b^\alpha = 0$. Proceeding in the same manner one can derive the other structure relations.

There is a nice way to generate all these relations systematically using the language of *NQ-manifolds*. Let us recall the basic definitions [21], [22], [23], [24]. An *N-manifold* is a non-negatively integer graded supermanifold, whose N-grading is compatible with the underlying \mathbb{Z}_2 -grading (Grassman parity). In other words, an N-manifold is just a supermanifold with a privileged class of atlases in which particular coordinates are assigned non-negative integer degrees (even coordinates have even degrees, while odd ones have odd degrees) so that the changes of coordinates respect these degrees. The highest degree of coordinates is called the *degree* of an N-manifold. For example, if $\deg M = 0$, then M is just an ordinary manifold. Finally, an *NQ-manifold* is an N-manifold endowed with integrable vector field Q of degree 1, called a *homological vector field*. Since Q is odd, the integrability condition $[Q, Q] = 2Q^2 = 0$ is nontrivial. The classical example of an NQ-manifold of degree 1 is the anti-cotangent bundle ΠTM (cotangent bundle with the reverse parity of fibers). The functions on ΠTM are just

³This technical restriction can be weakened.

inhomogeneous differential forms on M and Q is the usual exterior differential. More generally, an NQ-manifold of degree 1 is the same as Lie algebroid. For this reason it is natural to name the NQ-manifolds of degree n as *n-algebroids* [24].

A general homological vector field looks like⁴

$$(5.14) \quad Q = c^{\alpha_1} X_{\alpha_1}^i(x) \frac{\partial}{\partial x^i} + \sum_{k=1}^{\deg \mathcal{M}} c^{\alpha_{k+1}} d_{\alpha_{k+1}}^{\alpha_k}(x) \frac{\partial}{\partial c^{\alpha_k}} + \dots,$$

where $\deg(x^i) = 0$, $\deg(c^{\alpha_k}) = k$, and dots stand for higher orders in the positively graded variables c^{α_k} . Evaluating the equation $Q^2 = 0$ at the first order in c 's one recovers the cochain complex axioms $d_{\alpha_m}^{\alpha_{m-1}} d_{\alpha_{m+1}}^{\alpha_m} = 0$, the second order in c^{α_1} reproduces the involution relations (5.1), Rel. (5.9) contributes to the cubic order, and so on⁵. Thus, we see that the resolution (5.6) for the involutive distribution $X : \mathcal{E} \rightarrow TM$ is just a regular n -algebroid.

Although the language of NQ-manifolds is quite convenient to describe the structure of n -algebroids as such, it becomes inappropriate when one tries to incorporate the symplectic structure entering the factorization (2.20). Here we would like to present a new geometric framework providing a uniform description for both n -algebroid and symplectic structures underlying factorizable Poisson brackets. For the sake of simplicity we restrict our consideration to the case of 2-algebroids. The general construction will be developed elsewhere. Before going into details let us give two examples which are of interest by themselves.

Example 1. (Poisson-Lie algebras.) Consider an invariant Poisson bracket on a Lie group G associated with the bi-vector

$$(5.15) \quad \alpha = r^{ij}(L_i \wedge L_j - R_i \wedge R_j).$$

Here L_i, R_j are left and right invariant vector fields on G , and the matrix (r^{ij}) obeys the Yang-Baxter equation

$$(5.16) \quad f_{ml}^i (r^{jn} f_{ns}^l r^{sk} + \text{cycle}(j, l, k)) + \text{cycle}(i, j, k) = 0,$$

⁴All derivatives are assumed to be acting on the left.

⁵Notice, that any n -algebroid can also be viewed as an $(n+1)$ -algebroid whose higher structure functions just equal to zero.

f_{ij}^k being the structure constants of the corresponding Lie algebra $L(G)$. If $\det(r^{ij}) \neq 0$, we have the Poisson bi-vector α associated with the symplectic structure r^{ij} and the 2-algebroid

$$(5.17) \quad 0 \leftarrow TG \xleftarrow{(L,R)} TG \oplus TG \xleftarrow{d_2} TG \leftarrow 0,$$

where $d_2 = (1, A)$, and A is the automorphism of the tangent bundle TG relating the left and right invariant vector fields: $L_i = A_i^j R_j$.

Example 2. (Universal factorization.) Any Poisson bi-vector $\alpha = \alpha^{ij} \partial_i \wedge \partial_j$ can be factorized in a skew-symmetric product of Hamiltonian and coordinate vector fields:

$$(5.18) \quad \alpha = P_i \wedge Q^i, \quad P_i = \partial_i, \quad Q^i = \alpha^{ij} \partial_j.$$

The local vector distribution (P_j, Q^j) is obviously transitive and hence involutive. Moreover, there is a one-parameter ambiguity in writing the involution relations:

$$(5.19) \quad \begin{aligned} [P_i, P_j] &= 0, \\ [P_i, Q^j] &= \partial_i \alpha^{jk} P_k, \\ [Q^i, Q^j] &= t \partial_k \alpha^{ij} Q^k + (1-t) \partial_k \alpha^{ij} \alpha^{kn} P_n, \quad t \in \mathbb{R}. \end{aligned}$$

This ambiguity is due to linear dependence of the local vector fields:

$$(5.20) \quad Q^i = \alpha^{ij} P_j.$$

The last equations are already independent and we arrive at the following cochain complex

$$(5.21) \quad 0 \leftarrow TM \xleftarrow{(P,Q)} TM \oplus T^*M \xleftarrow{(1,\alpha)} TM \leftarrow 0,$$

which is exact provided α is non-degenerate on an everywhere dense domain in M .

Consider now a general NQ-manifold \mathcal{M} of degree 2. As for usual manifolds, the structure of \mathcal{M} can be described in terms of coordinate charts and transition functions gluing together individual N-graded domains $U \in \mathcal{M}$. Without loss of generality we can assume that each U is given by a direct product $\mathcal{U} \times \mathbb{R}_1^n \times \mathbb{R}_2^m$, where $\mathcal{U} \in M$ is an open contractible domain on the base manifold with local coordinates x^i , \mathbb{R}_1^n and \mathbb{R}_2^m are vector spaces with linear coordinates c^α and c^a , respectively. We set $\deg(x^i) = 0$, $\deg(c^\alpha) = 1$, $\deg(c^a) = 2$, so that x^i and c^a

are commuting, while c^α are anticommuting coordinates on $U \in \mathcal{M}$. If now U and U' are two graded domains with nonempty intersection, then the most general form of transition functions, compatible with the N-grading, is given by

$$(5.22) \quad x'^i = f^i(x), \quad c'^\alpha = A_\beta^\alpha(x)c^\beta, \quad c'^a = B_b^a(x)c^b + \frac{1}{2}F_{\alpha\beta}^a(x)c^\alpha c^\beta,$$

$f^i, A_\beta^\alpha, B_b^a$ being smooth functions on $\mathcal{U} \cap \mathcal{U}'$. The first equation defines transformation of local coordinates on the base manifold M . Disregarding the F -term, we see that the second and third equations are similar to those defining transition functions for (graded) vector bundles. Moreover, the matrix-valued functions A and B do really obey the standard cocycle conditions on overlaps of two and three coordinate charts, defining thus direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$ of two graded vector bundles.

In terms of local coordinates the most general homological vector field on \mathcal{M} reads

$$(5.23) \quad Q = c^\alpha X_\alpha^i \frac{\partial}{\partial x^i} + c^a d_a^\alpha \frac{\partial}{\partial c^\alpha} + \frac{1}{2} c^\beta c^\alpha f_{\alpha\beta}^\gamma \frac{\partial}{\partial c^\gamma} + c^\alpha c^a f_{\alpha a}^b \frac{\partial}{\partial c^b} + c^\gamma c^\beta c^\alpha f_{\alpha\beta\gamma}^a \frac{\partial}{\partial c^a}.$$

Using relations (5.22) one can derive transformation properties for the structure functions $X_\alpha^i, d_a^\gamma, f_{\alpha\beta}^\gamma, f_{a\beta}^b, f_{\alpha\beta\gamma}^a \in C^\infty(\mathcal{U})$ under coordinate changes. In particular, the F -term induces the shift

$$(5.24) \quad f_{\alpha\beta}^\gamma \rightarrow f_{\alpha\beta}^\gamma + F_{\alpha\beta}^a d_a^\gamma,$$

which reflects an inherent ambiguity concerning the choice of the structure functions (5.1) whenever X_α are linearly dependent. Also, it is not hard to see that X_α^i and d_a^γ transform homogeneously, i.e., as sections of the associated vector bundles $\mathcal{E}_1^* \oplus TM$ and $\mathcal{E}_2^* \oplus \mathcal{E}_1$.

Now suppose that \mathcal{M} defines a 2-algebroid factorizing a Poisson bi-vector α . Our aim is to give a unified description for both the 2-algebroid and the symplectic structure entering this factorization. It turns out that all structure relations underlying the factorization (5.2) can be described in terms of an abelian connection (covariant derivative) acting on a bundle of odd Poisson algebras over M . The construction goes as follows.

Let $E_0 \oplus E_1$ be a \mathbb{Z}_2 -graded vector bundle over M defined by aforementioned gluing cocycles A and $(B^{-1})^*$, that is $E_1 = \mathcal{E}_1$, $E_0 = \mathcal{E}_2^*$. If c^α and π_a are linear coordinates in the fibers of E_1 and E_0 over a trivializing domain $\mathcal{U} \in M$, we set $\epsilon(c^\alpha) = 1$, $\epsilon(\pi_a) = \epsilon(x^i) = 0$. It is

convenient to think of this bundle as a formal supermanifold \mathcal{N} with even coordinates x^i, π_a and odd coordinates c^α . The word “formal” reflects the fact that we allow the functions on \mathcal{N} to be given by formal power series in π ’s. These functions form a supercommutative algebra \mathcal{F} with the generic element

$$(5.25) \quad f(x, c, \pi) = \sum_{k,n} f_{\alpha_1 \dots \alpha_n}^{a_1 \dots a_k}(x) \pi_{a_1} \dots \pi_{a_k} c^{\alpha_1} \dots c^{\alpha_n}.$$

The algebra $\mathcal{F} = \oplus \mathcal{F}_{n,m}$ is naturally bigraded w.r.t. powers of c ’s and π ’s and is isomorphic to the tensor algebra of sections of the associated vector bundle $S^\bullet E_0^* \otimes \Lambda^\bullet E_1^*$.

The space \mathcal{F} can also be endowed with the structure of odd Poisson algebra. To this end, we introduce the odd Laplacian $\Delta : \mathcal{F}_{m,n} \rightarrow \mathcal{F}_{m-1,n-1}$:

$$(5.26) \quad \Delta f = d_a^\alpha(x) \frac{\partial^2 f}{\partial c^\alpha \partial \pi_a}.$$

Clearly, $\Delta^2 = 0$. The odd Poisson bracket $(\cdot, \cdot) : \mathcal{F}_{n,m} \otimes \mathcal{F}_{k,l} \rightarrow \mathcal{F}_{n+k-1,m+l-1}$ is defined by the rule:

$$(5.27) \quad (-1)^{\epsilon(f)}(f, g) = \Delta(f \cdot g) - \Delta f \cdot g - (-1)^{\epsilon(f)} f \cdot \Delta g.$$

It obeys the standard identities which may be taken as the axioms of an odd Poisson manifold:

$$(5.28) \quad \begin{aligned} \epsilon(f, g) &= \epsilon(f) + \epsilon(g) + 1 && (\text{mod } 2), \\ (f, g) &= -(g, f)(-1)^{(\epsilon(f)+1)(\epsilon(g)+1)} && (\text{symmetry}), \\ (f, gh) &= (f, g)h + (f, h)g(-1)^{\epsilon(g)\epsilon(h)} && (\text{Leibnitz rule}), \\ (f, (g, h))(-1)^{(\epsilon(f)+1)(\epsilon(h)+1)} + \text{cycle}(f, g, h) &= 0 && (\text{Jacobi identity}). \end{aligned}$$

Notice that Δ respects the odd Poisson bracket (5.27) in the sense that

$$(5.29) \quad \Delta(f, g) = (\Delta f, g) + (-1)^{\epsilon(f)+1}(f, \Delta g).$$

The algebra \mathcal{F} contains a special element $\omega = \frac{1}{2}\omega_{\alpha\beta}c^\alpha c^\beta \in \mathcal{F}_{2,0}$ associated with the symplectic structure. The adjoint action of ω gives rise to the nilpotent differentiation $\delta : \mathcal{F}_{n,m} \rightarrow \mathcal{F}_{n+1,m-1}$:

$$(5.30) \quad \delta f = (\omega, f) = -c^\alpha \omega_{\alpha\beta} d_a^\beta \frac{\partial f}{\partial \pi_a}, \quad \delta^2 = 0.$$

It easy to see that the δ -cohomology is trivial when evaluated on $\mathcal{F}_{\bullet,k}$ with $k > 0$.

Now we would like to endow the bundle of odd Poisson algebras \mathcal{F} with a sort of partial connection $\nabla : \mathcal{F}_{m,n} \rightarrow \mathcal{F}_{m+1,n}$ making possible parallel transport along the leaves of the anchor foliation $F(M)$. Treating ∇ as an odd vector field on \mathcal{N} , we set

$$(5.31) \quad \nabla a = c^\alpha \left(X_\alpha^i \frac{\partial}{\partial x^i} + \frac{1}{2} c^\beta f_{\alpha\beta}^\gamma \frac{\partial}{\partial c^\gamma} + \pi_a f_{\alpha b}^a \frac{\partial}{\partial \pi_b} \right) a + (W_1, a),$$

where the structure functions X_α^i , $f_{\alpha\beta}^\gamma$, $f_{\alpha b}^a$ are the same as in Eq. (5.23) and $W_1 = c^\alpha c^\beta W_{\alpha\beta}^a \pi_a \in \mathcal{F}_{2,1}$ is given by the r.h.s. of Eq. (5.11). Using the definition of ∇ , one can rewrite Eq. (5.11) as

$$(5.32) \quad \nabla \omega = 0$$

or, equivalently,

$$(5.33) \quad \nabla \delta + \delta \nabla = 0.$$

The main property of the local vector field ∇ is that it respects the odd Poisson bracket, i.e.,

$$(5.34) \quad \nabla(f, g) = -(\nabla f, g) + (-1)^{\epsilon(f)+1} (f, \nabla g),$$

for any $f, g \in \mathcal{F}|_U$. Squaring ∇ , we get an internal derivation of the odd Poisson algebra:

$$(5.35) \quad \nabla^2 f = (R, f),$$

where one can thought of the odd function $R = c^\gamma c^\beta c^\alpha f_{\alpha\beta\gamma}^a \pi_a \in \mathcal{F}_{3,1}$ as the curvature of ∇ .

Like a curvature, R obeys the Bianchi identity

$$(5.36) \quad \nabla R = 0.$$

Now we can extend ∇ from a local coordinate chart U to the whole \mathcal{N} . To this end, we choose a trivializing covering $\{\mathcal{U}_i\}$ of M together with local connections ∇_i on $(E_0 \oplus E_1)|_{\mathcal{U}_i}$. It follows from (5.32) that on each nonempty intersection $\mathcal{U}_i \cap \mathcal{U}_j$

$$(5.37) \quad \nabla_i - \nabla_j = (\delta\phi_{ij}, \cdot),$$

for some $\phi_{ij} \in \mathcal{F}_{1,2}|_{\mathcal{U}_i \cap \mathcal{U}_j}$. Then, on each nonempty intersection $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$ the functions ϕ_{ij} satisfy the relation

$$(5.38) \quad \delta(\phi_{ij} + \phi_{jk} + \phi_{ki}) = 0.$$

Since the δ -cohomology is trivial on $\mathcal{F}_{1,2}$ we conclude that

$$(5.39) \quad \phi_{ij} + \phi_{jk} + \phi_{ki} = \delta\psi_{ijk} \quad \text{on } \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k,$$

for some $\chi_{ijk} \in \mathcal{F}_{0,3}|_{\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k}$. Again, from the last equation it follows that on each nonempty intersection $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_l$ of four domains one has

$$(5.40) \quad \chi_{ijk} - \chi_{jkl} + \chi_{kli} - \chi_{lij} = 0.$$

Notice that Eq. (5.37) does not define ϕ_{ij} uniquely as we are free to add to them any δ -closed terms $\delta\psi_{ij} \in \mathcal{F}_{1,2}$. This modifies the r.h.s. of Eq. (5.39) as follows:

$$(5.41) \quad \chi_{ijk} \longrightarrow \chi_{ijk} - (\psi_{ij} + \psi_{jk} + \psi_{ki}).$$

Eqs. (5.40), (5.41) imply that to any collection of local connections ∇_i we have associated an element χ of the second Čech cohomology group with coefficients in $\mathcal{F}_{0,3}$. Since this group is clearly isomorphic to the second De Rham's cohomology group of M we can think of χ as an element of $H^2(M)$.

Given a partition of unity $\{h^i\}$ subordinated to the covering $\{\mathcal{U}_i\}$, we set

$$(5.42) \quad \phi_i = \phi_{ij} h^j.$$

It is not hard to check, using Rel. (5.39), that the new local connections $\nabla'_i = \nabla_i - (\delta\phi_i, \cdot)$ already coincide on each intersection $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$. Thus, there are no topological obstructions to introducing a partial connection of the form (5.31) and we can regard $\chi \in H^2(M)$ as an invariant of ∇ .

Combining now the action of ∇ with internal differentiations of the odd Poisson algebra \mathcal{F} , one can construct a more general connection $D : \mathcal{F}_{n,\bullet} \rightarrow \mathcal{F}_{n+1,\bullet}$:

$$(5.43) \quad Da = \delta a + \nabla a - (W, a) = \nabla a + (\omega - W, a),$$

W being an element of $\mathcal{F}_{2,\bullet}$. if $D^2 = 0$, we refer to D as an abelian connection. The condition of D to be an abelian connection is equivalent to the following equation:

$$(5.44) \quad \delta W = R + \nabla W + \frac{1}{2}(W, W).$$

The existence of an abelian connection follows from the solubility of (5.44). Indeed, substituting expansion

$$(5.45) \quad W = \sum_{k=2}^{\infty} W_k, \quad W_k \in \mathcal{F}_{2,k},$$

into (5.44) one gets

$$(5.46) \quad \begin{aligned} \delta W_2 &= R, \\ \delta W_{n+1} &= \nabla W_n + \frac{1}{2} \sum_{k=2}^n (W_{n-k+2}, W_k), \quad n \geq 2. \end{aligned}$$

Since the δ -cohomology is trivial when evaluated on $\mathcal{F}_{2,k}$ with $k > 0$, the first equation is soluble provided $\delta R = 0$. But the last condition immediately follows from the identities $0 = \nabla^2 \omega = (R, \omega) = -\delta R$. Proceeding by induction, one can see that the r.h.s. of the $(n+1)$ -th equation is δ -closed (and thus is δ -exact) provided all the previous equations for W_2, \dots, W_n have been satisfied.

The main results of this section can be summarized as follows.

Proposition 5.1. *Suppose we are given by the following data:*

- (1) *a short exact sequence*

$$0 \rightarrow \mathcal{E}_2 \xrightarrow{d} \mathcal{E}_1 \xrightarrow{X} F \rightarrow 0,$$

where $\mathcal{E}_1 \rightarrow M$, $\mathcal{E}_2 \rightarrow M$ are vector bundles over a smooth manifold M , F is an integrable subbundle of the tangent bundle TM , and d, X are M -morphisms of the vector bundles (not necessarily of constant rank);

- (2) *a non-degenerate, skew-symmetric, bilinear form ω on \mathcal{E}_1 inducing a Poisson bi-vector field on the base manifold:*

$$\alpha = \langle \omega, X \wedge X \rangle \in \wedge^2 TM, \quad [\alpha, \alpha] = 0.$$

(Here we identify $X : \mathcal{E}_1 \rightarrow F \subset TM$ with a section of $\mathcal{E}_1^ \otimes TM$.)*

Then to each set of such data one can associate

- (1) an invariant χ taking value in the second group of De Rham's cohomology of M , and
- (2) a bundle of odd Poisson algebras \mathcal{F} over M together with an abelian connection D differentiating \mathcal{F} such that the condition $D^2 = 0$ generates all structure relations arising from the integrability of F and the Jacobi identity for α .

The generating procedure stated above could be viewed as starting point for quantizing general factorizable brackets associated with symplectic 2-algebroids along the lines of Sections 3, 4.

REFERENCES

- [1] F.A. Berezin, “Quantization”, Izv. Akad. Nauk. **38** (1974) 1116-1175; “General concept of quantization”, Comm. Math. Phys. **40** (1975) 153-174.
- [2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, “Deformation Theory and Quantization. 1. Deformation of Symplectic Structures”, Ann. Phys.(N.Y.) **111** (1978), 61.
- [3] B.V. Fedosov, “A Simple Geometrical Construction of Deformation Quantization”, J. Diff. Geom. **40** (1994) 213.
- [4] B.V. Fedosov, Deformation Quantization and Index Theory, Berlin, Germany: Akademie-Verl. (1996) (Mathematical Topics: 9).
- [5] M. Kontsevich, “Deformation quantization of Poisson manifolds, I”, Lett. Math. Phys. **66** (2003) 157.
- [6] A. S. Cattaneo, G. Felder, L. Tomassini, “From local to global deformation quantization of Poisson manifolds”, Duke Math. J. **115** (2002) 329.
- [7] V. Dolgushev, “Covariant and Equivariant Formality Theorem”, Adv. Math. **191** (2005) 147-177.
- [8] A.S. Cattaneo and G. Felder, “A path integral approach to the Kontsevich quantization formula”, Commun. Math. Phys. **212** (2000) 591.
- [9] I.A. Batalin and G.A. Vilkovisky, “Gauge algebra and quantization”, Phys. Lett. **B102** (1981) 27-31.
- [10] A. Alekseev and Y. Kosmann-Schwarzbach, “Manin pairs and moment maps”, math.DG/9909176.
- [11] R. Nest and B. Tsygan, “Deformations of symplectic Lie algebroids, deformations of holomorphic symplectic structures, and index theorems”, Asian J. Math. **5** (2001) 599-635.
- [12] I. Vaisman, “Fedosov Quantization on Symplectic Ringed Spaces”, math.SG/0106070.
- [13] M. A. Rieffel, “Deformation Quantization for Action of R^d ”, Memoirs A.M.S. **506**, AMS, Providence, 1993.
- [14] P. Xu, “Triangular dynamical r -matrices and quantization”, Adv. Math. **166**, 1 (2002) 1-49.

- [15] V.A. Dolgushev, A.P. Isaev, S.L. Lyakhovich and A.A. Sharapov, “*On the Fedosov deformation quantization beyond the regular Poisson manifolds*”, Nucl.Phys. **B645** (2002) 457-476.
- [16] E.S. Fradkin and G.A. Vilkovisky, “*Quantization of relativistic systems with constraints*”, Phys. Lett. **B55** (1975) 224-226; I.A. Batalin and G.A. Vilkovisky, “*Relativistic S-matrix of dynamical systems with boson and fermion constraints*”, Phys. Lett. **B69** (1977) 309-312.
- [17] I.A. Batalin and E.S. Fradkin, “*A generalized canonical formalism and quantization of reducible gauge theories*”, Phys.Lett. **B122** (1983) 157-164.
- [18] M. Henneaux and C. Teitelboim, Quantization of Gauge Systems (Princeton U.P., NJ, 1992).
- [19] M.A. Grigoriev and S.L. Lyakhovich, “*Fedosov Deformation Quantization as a BRST Theory*”, Commun. Math. Phys. **218** (2001) 437.
- [20] I.A. Batalin, M.A. Grigoriev, S.L. Lyakhovich, “*Star Product for Second Class Constraint Systems from a BRST Theory*”, Theor. Math. Phys. **128** (2001) 1109-1139; hep-th/0101089.
- [21] A. Vaintrob, “*Lie algebroids and homological vector fields*”, Uspekhi Mat. Nauk, **52** (1997) 428-429.
- [22] P. Ševera, “*Some title containing the words “homotopy” and “symplectic”, e.g. this one*”, math.SG/0105080.
- [23] Th. Th. Voronov, “*Graded manifolds and Drinfeld doubles for Lie bialgebroids*”, Contemp. Math. **315** (2002) 131-168.
- [24] D. Roytenberg, “*On the structure of graded symplectic supermanifolds and Courant algebroids*”, Contemp. Math. **315** (2002) 169-185.
- [25] A. Cannas da Silva and A. Weinstein, Geometric Models for Noncommutative Algebras, *Berkeley Mathematics Lecture Notes*, vol.10 (AMS, Providence, RI, 1999).
- [26] H. Sussman, “*Orbits of families of vector fields and integrability of distributions*”, Trans. Amer. Math. Soc. **180**, 171-188.
- [27] K.C.H. Mackenzie and P. Xu, “*Lie bialgebroids and Poisson groupoids*”, Duke Math. J. **73** (1994) 415-452.
- [28] A.G. Elashvili, “*Frobenius Lie algebras*”, Funct. Anal. Applic. **16** (1982) 94-95.
- [29] I. A. Batalin and I. V. Tyutin, “*Existence Theorem For The Effective Gauge Algebra In The Generalized Canonical Formalism With Abelian Conversion Of Second Class Constraints*”, Int.J. Mod. Phys. A **6** (1991) 3255.
- [30] T. Courant, *Dirac manifolds*, Trans. Amer. Math. Soc. **319** (1990) 631-661.
- [31] C. Emrich and A. Weinstein, “*The differential geometry of Fedosov’s quantization*”, Lie Theory and Geometry, 217-239, Progr. Math. **123**, Birkhäuser Boston, Boston, 1994.

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